

# Quantum product on the big phase space and the Virasoro conjecture

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Quantum cohomology is a family of new ring structures on the space of cohomology classes of a compact symplectic manifold (or a smooth projective variety)  $V$ . The quantum products are defined by third order partial derivatives of the generating function of primary Gromov-Witten invariants of  $V$  (cf. [RT1]). In a similar way, using the generating function of all descendant Gromov-Witten invariants, we can define products on an infinite dimensional vector space, called the big phase space, which can be thought of as a product of infinite copies of the small phase space  $H^*(V; \mathbb{C})$ . It seems that the products on the big phase space have not gotten enough attention in the literature so far. In this paper we will study some basic structures of such products and apply them to the study of topological recursion relations and the Virasoro conjecture.

The Virasoro conjecture predicts that the generating function of the Gromov-Witten invariants is annihilated by infinitely many differential operators, denoted by  $\{L_n \mid n \geq -1\}$ , which form a half branch of the Virasoro algebra. This conjecture was proposed by Eguchi, Hori and Xiong [EHX] and also by S. Katz (cf. [CK] [EJX]). It is a natural generalization of a conjecture of Witten (cf. [W2] [Ko] [W2]) and provides a powerful tool in the computation of Gromov-Witten invariants. The genus-0 Virasoro conjecture was proved in [LT] (cf. [DZ2] and [G2] for alternative proofs). The genus-1 Virasoro conjecture for manifolds with semisimple quantum cohomology was proved in [DZ2]. Without assuming semisimplicity, the genus-1 Virasoro conjecture was reduced to the genus-1  $L_1$ -constraint on the small phase space in [L1]. It was also proved in [L1] [L2] that the genus-1 Virasoro conjecture holds if the quantum cohomology is not too degenerate (a condition weaker than semisimplicity).

The genus- $g$  Virasoro conjecture with  $g \geq 1$  can be formulated in a way which computes the derivatives of the genus- $g$  generating function along a sequence of vector fields, called the Virasoro vector fields (see Section 5.1). The study of the properties of these vector fields will be important in both proving and applying the Virasoro conjecture in all genera. In this paper we will give a simple recursive description of the Virasoro vector fields (see equation (41) and Theorem 4.7). This recursive description enables us to understand the relations between the Virasoro vector fields and the quantum powers of the Euler vector field defined by equation (27). The action of the Virasoro vector fields on the generating function of genus- $g$  Gromov-Witten invariants is equivalent to the action of a

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sequence of vector fields constructed from quantum powers of the Euler vector field. To prove this fact, we need to use the quantum product on the big phase space to reinterpret genus- $g$  topological recursion relations. The most important difference between the quantum product on the big phase space and the one on the small phase space is that there is no identity element for the product on the big phase space. The best candidate for the identity is the string vector field defined by equation (1). However, this vector field is not an identity in the usual sense. How close this vector field is to an identity is reflected through various topological recursion relations. Such interpretation of topological recursion relations will be very useful in the study of Virasoro conjecture. For example, this will enable us to represent all the vector fields obtained from the Virasoro vector fields under certain naturally defined operations in terms of twisted quantum powers of the Euler vector field (cf. Theorem 4.8 and the comments afterwards). This explains why Virasoro constraints are very powerful for manifolds with semisimple quantum cohomology, as in this case the quantum powers of the Euler vector field span the space of primary vector fields.

We believe that the structures defined in this paper will be very useful in the study of the Virasoro conjecture for all genera. As a demonstration, we will apply these structures to the genus-2 Virasoro conjecture. The study of the genus-2 Virasoro conjecture is important because this is the first case that we do not have a formula to reduce the problem to the small phase space. The behavior of the Virasoro conjecture in this case will provide much needed insight in what we should expect in the higher genus cases. Moreover the techniques developed here could be easily adapted to the study of higher genus Virasoro conjecture. In this paper, we will prove that for any manifold, the genus-2 Virasoro conjecture holds if and only if the genus-1 and genus-2  $L_1$ -constraints hold (see Theorem 5.9). The main reason for this result is that for  $n \geq 2$ , the genus-2 part of the genus-2  $L_n$ -constraint can be recursively computed from the genus-2 part of the genus-2  $L_1$ -constraint (See Corollary 5.12 for the precise recursion formula). In the case that the quantum cohomology of the underlying manifold is not too degenerate (in particular is semisimple), such recursion formula also uniquely determines the genus-2 part of  $L_1$ -constraint (see Theorem 5.17). Therefore the genus-2 Virasoro conjecture for such manifolds can now be reduced to a genus-1 problem. To complete the proof of the genus-2 Virasoro conjecture for this case would involve more detailed analysis of the complicated tensors  $A_1$ ,  $A_2$  and  $B$  used in equations (19) - (21). We will do this in a separate paper. We notice that the first 3 Virasoro operators  $\{L_{-1}, L_0, L_1\}$  form a 3-dimensional subalgebra of the Virasoro algebra which is isomorphic to  $sl(2)$ . So Theorem 5.9 in particular implies that for any manifold, the  $sl(2)$  symmetry of the genus-2 generating function is sufficient to deduce the genus-2 Virasoro conjecture. Similar situation also occurred in the genus-1 case (cf. [L1]). We wonder whether the same pattern will continue for all genera.

The major application of the genus- $g$  Virasoro conjecture is to compute the genus- $g$  generating function of the Gromov-Witten invariants in terms of data with genus less than  $g$ . Although in principle this can be done if the quantum cohomology is semisimple, it is not easy to solve the genus- $g$  generating function explicitly from the Virasoro constraints for general manifolds with semisimple quantum cohomology. Recently Dubrovin-Zhang

and Eguchi-Getzler-Xiong computed the genus-2 generating function for Frobenius manifolds with two primary fields assuming that the genus-2 Virasoro constraints hold (cf. [EGX]). Note that in the Gromov-Witten theory, only  $\mathbb{C}P^1$  has two primary fields. In this paper, we will prove (without assuming the Virasoro constraints) that the genus-2 generating function can be expressed explicitly in terms of genus-0 and genus-1 data if the quantum cohomology of the underlying manifold is not too degenerate (in particular, if the quantum cohomology is semisimple) (see Theorem 5.17). As proved in [DZ1], the genus-1 generating function can be expressed explicitly in terms of genus-0 data when the quantum cohomology is semisimple. Therefore Theorem 5.17 also implies that the genus-2 generating function can be expressed explicitly in terms of genus-0 data in the semisimple case.

It will be interesting to see how many ingredients are needed in the study of the genus-2 Virasoro conjecture. For this purpose, we need to study relations among the genus-2 topological recursion relations in [G1] and [BP] (see equations (16) - (18)). In this paper, we will show that equation (17) implies equation (16) (see Theorems 2.6), and equation (16) together with equation (18) also implies equation (17) (see Theorem 2.9). This tells us that, at least for manifolds whose quantum cohomology is not too degenerate, the number of ingredients needed in the study of the genus-2 Virasoro conjecture is the same as that for the genus-1 case. It will be interesting to investigate whether this phenomenon will continue in higher genus cases.

This paper is organized as follows. In Section 1, we define the quantum product on the big phase space. In Section 2, we first re-formulate topological recursion relations using an operator  $T$  which measures the difference between the string vector field and an identity of the quantum product, and then apply it to study relations among genus-2 topological recursion relations. In Section 3, we study properties of quantum powers of the Euler vector field. Most properties of these vector fields can be derived from the quasi-homogeneity equation (28). In particular, the recursive operator  $R$  used to describe the Virasoro vector fields arrive naturally in the study of the quasi-homogeneity equation (see Theorem 3.9). Section 4 is devoted to the study of properties of the Virasoro vector fields, in particular their relation with quantum powers of the Euler vector field. Using the operator  $T$  and its right inverse, we can produce some Lie algebras which contains the Lie algebra of Virasoro vector fields as a proper subalgebra. In particular, we will give a realization of the Lie algebra of integral pseudo-differential operators on the unit circle. In Section 5, we first formulate the Virasoro conjecture using recursive operators, and then study the genus-2 Virasoro conjecture. For manifolds whose quantum cohomology is not too degenerate, we explicitly solve the genus-2 generating function in terms of genus-0 and genus-1 data in Section 5.4. In the proof of Theorem 5.17, we need a lemma which is proven in the appendix.

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# 1 Quantum Product on the big phase space

For simplicity, we assume that  $V$  is a smooth projective variety with  $H^{\text{odd}}(V; \mathbb{C}) = 0$ . All results in this paper are also true for compact symplectic manifolds except those in Section 5.2. Choose a basis  $\{\gamma_1, \gamma_2, \dots, \gamma_N\}$  of  $H^*(V; \mathbb{C})$  with  $\gamma_1$  equal to the identity of the ordinary cohomology ring. Let

$$\langle \tau_{n_1}(\gamma_{\alpha_1}) \tau_{n_2}(\gamma_{\alpha_2}) \dots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_g$$

be the genus- $g$  descendant Gromov-Witten invariant associated to  $\gamma_{\alpha_1}, \dots, \gamma_{\alpha_k}$  and non-negative integers  $n_1, \dots, n_k$  (which represent the powers of the first Chern classes of certain tautological line bundles over the moduli space of stable maps from genus- $g$  curves to  $V$  with  $k$  marked points) (cf. [W1] [RT2]). The genus- $g$  generating function is defined to be

$$F_g = \sum_{k \geq 0} \frac{1}{k!} \sum_{\substack{\alpha_1, \dots, \alpha_k \\ n_1, \dots, n_k}} t_{n_1}^{\alpha_1} \dots t_{n_k}^{\alpha_k} \langle \tau_{n_1}(\gamma_{\alpha_1}) \tau_{n_2}(\gamma_{\alpha_2}) \dots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_g,$$

where  $\{t_n^\alpha \mid n \in \mathbb{Z}_+, \alpha = 1, \dots, N\}$  is an infinite set of parameters. We can think of these parameters as coordinates on an infinite dimensional vector space, called the *big phase space*. The finite dimensional subspace defined by  $\{t_n^\alpha = 0 \text{ if } n > 0\}$  is called the *small phase space*. The function  $F_g$  is understood as a formal power series of  $t_n^\alpha$ . As in [LT], it is convenient to introduce a  $k$ -tensor  $\langle \underbrace{\dots}_k \rangle$  defined by

$$\langle \mathcal{W}_1 \mathcal{W}_2 \dots \mathcal{W}_k \rangle_g := \sum_{m_1, \alpha_1, \dots, m_k, \alpha_k} f_{m_1, \alpha_1}^1 \dots f_{m_k, \alpha_k}^k \frac{\partial^k}{\partial t_{m_1}^{\alpha_1} \partial t_{m_k}^{\alpha_k}} F_g,$$

for (formal) vector fields  $\mathcal{W}_i = \sum_{m, \alpha} f_{m, \alpha}^i \frac{\partial}{\partial t_m^\alpha}$  where  $f_{m, \alpha}^i$  are (formal) functions on the big phase space. We can also view this tensor as the  $k$ -th covariant derivative of  $F_g$ . This tensor is called the *k-point (correlation) function*. We will always identify  $\tau_n(\gamma_\alpha)$  with the tangent vector field  $\frac{\partial}{\partial t_n^\alpha}$  on the big phase space and abbreviate  $\tau_0(\gamma_\alpha)$  as  $\gamma_\alpha$ . We also consider  $\tau_n(\gamma_\alpha)$  with  $n < 0$  as a zero vector field. Let  $\eta_{\alpha\beta} = \int_V \gamma_\alpha \cup \gamma_\beta$  be the Poincare pairing. We use  $\eta = (\eta_{\alpha\beta})$  and its inverse  $\eta^{-1} = (\eta^{\alpha\beta})$  to lower and raise indices. So  $\tau_n(\gamma^\alpha) = \eta^{\alpha\beta} \tau_n(\gamma_\beta)$ . Here we adopt the convention of summing over repeated indices. We call a vector field  $\mathcal{W} = \sum_{m, \alpha} f_{m, \alpha} \tau_m(\gamma_\alpha)$  a *primary vector field* if  $f_{m, \alpha} = 0$  whenever  $m > 0$ , a *descendant vector field* if  $f_{m, \alpha} = 0$  whenever  $m = 0$ .

For any two vector fields  $\mathcal{U}$  and  $\mathcal{W}$  on the big phase, define the **quantum product** of  $\mathcal{U}$  and  $\mathcal{W}$  by

$$\mathcal{U} \bullet \mathcal{W} := \langle \mathcal{U} \mathcal{W} \gamma^\alpha \rangle_0 \gamma_\alpha.$$

By definition, the quantum product of two vector fields is always a primary vector field. This product is apparently commutative. It is also associative due to the *generalized WDVV equation*

$$\langle \mathcal{W}_1 \mathcal{W}_2 \gamma^\alpha \rangle_0 \langle \gamma_\alpha \mathcal{W}_3 \mathcal{W}_4 \rangle_0 = \langle \mathcal{W}_1 \mathcal{W}_3 \gamma^\alpha \rangle_0 \langle \gamma_\alpha \mathcal{W}_2 \mathcal{W}_4 \rangle_0,$$

which follows in turn from the genus-0 topological recursion relation (cf. [W1]). When restricted to tangent vector fields on the small phase, this is precisely the product in the quantum cohomology of  $V$  (also called the big quantum cohomology by some authors). For any vector field  $\mathcal{W}$  on the big phase space, we define  $\mathcal{W}^k$  to be the  $k$ -th quantum power of  $\mathcal{W}$ . i.e.,

$$\mathcal{W}^k = \underbrace{\mathcal{W} \bullet \mathcal{W} \bullet \cdots \mathcal{W}}_k,$$

for  $k > 0$ .

For the quantum product on the small phase space, the constant vector field  $\gamma_1$ , which was chosen to be the identity of the ordinary cohomology ring, is also the identity for the quantum cohomology. However on the big phase space, there is no identity vector field for the quantum product. A vector field which is close to an identity is the *string vector field*

$$\mathcal{S} = - \sum_{m, \alpha} \tilde{t}_m^\alpha \tau_{m-1}(\gamma_\alpha), \quad (1)$$

where  $\tilde{t}_m^\alpha = t_m^\alpha - \delta_m^1 \delta_\alpha^1$ . This vector field can be considered as a sort of identity in the following sense:

- Lemma 1.1** (i)  $\mathcal{S} \bullet \mathcal{W} = \mathcal{W}$  for any primary vector field  $\mathcal{W}$ .  
(ii)  $\mathcal{S} \bullet \mathcal{U} \bullet \mathcal{W} = \mathcal{U} \bullet \mathcal{W}$  for all vector fields  $\mathcal{U}$  and  $\mathcal{W}$ .

**Proof:** This is a consequence of the *string equation*

$$\langle\langle \mathcal{S} \rangle\rangle_g = \frac{1}{2} \delta_{g,0} \eta_{\alpha\beta} t_0^\alpha t_0^\beta.$$

In fact, taking second derivatives of the genus-0 string equation, we obtain  $\langle\langle \mathcal{S} \gamma_\alpha \gamma_\beta \rangle\rangle_0 = \delta_\alpha^\beta$  for all  $\alpha$  and  $\beta$  (cf. [LT, Lemma 1.1]). This is equivalent to (i). The second formula follows from the associativity of the quantum product and (i) since  $\mathcal{U} \bullet \mathcal{W}$  is a primary vector field by definition.  $\square$

We define an equivalence relation between two vector fields on the big phase space:

**Definition 1.2** We say two vectors  $\mathcal{U}$  and  $\mathcal{W}$  are equivalent, denoted by  $\mathcal{U} \sim \mathcal{W}$ , if  $\mathcal{U} \bullet \mathcal{V} = \mathcal{W} \bullet \mathcal{V}$  for all vector fields  $\mathcal{V}$ .

The first formula in Lemma 1.1 implies that two primary vector fields are equivalent if and only if they are equal. The second formula in Lemma 1.1 is equivalent to

$$\mathcal{S} \bullet \mathcal{W} \sim \mathcal{W}$$

for any  $\mathcal{W}$ . Therefore we can say that  $\mathcal{S}$  is an identity in this weak sense. Since  $\mathcal{S} \bullet \mathcal{W}$  is always a primary vector field, this relation also tells us that any vector field on the big phase space is equivalent to a primary vector field. Moreover if  $\mathcal{U}$  is a primary vector field, then for any vector field  $\mathcal{W}$ ,

$$\mathcal{W} \sim \mathcal{U} \iff \mathcal{S} \bullet \mathcal{W} = \mathcal{U}. \quad (2)$$

For the convenience of later computations, we define

$$\overline{\mathcal{W}} := \mathcal{S} \bullet \mathcal{W} \quad (3)$$

for any vector field  $\mathcal{W}$ . In particular, Lemma 1.1 implies that  $\overline{\mathcal{S}}$  is the identity of the subalgebra of all primary vector fields with respect to the quantum product. Note that when restricted to the small phase space, both  $\mathcal{S}$  and  $\overline{\mathcal{S}}$  are equal to the identity of the ordinary (and the quantum) cohomology ring.

It is also convenient to introduce the following (functionally) linear transformations on the space of vector fields on the big phase space:

**Definition 1.3** For any vector field  $\mathcal{W} = \sum_{n,\alpha} f_{n,\alpha} \tau_n(\gamma_\alpha)$ , define

$$\begin{aligned} \tau_+(\mathcal{W}) &:= \sum_{n,\alpha} f_{n,\alpha} \tau_{n+1}(\gamma_\alpha), & \tau_-(\mathcal{W}) &:= \sum_{n,\alpha} f_{n,\alpha} \tau_{n-1}(\gamma_\alpha), \\ T(\mathcal{W}) &:= \tau_+(\mathcal{W}) - \langle\langle \mathcal{W} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha, & \pi(\mathcal{W}) &:= \sum_{\alpha} f_{0,\alpha} \gamma_\alpha. \end{aligned}$$

Then

$$\tau_+ \tau_-(\mathcal{W}) = \mathcal{W} - \pi(\mathcal{W}), \quad \tau_- \tau_+(\mathcal{W}) = \tau_- T(\mathcal{W}) = \mathcal{W}.$$

Moreover the genus-0 string equation implies that (cf. [LT, Lemma 1.1])

$$\overline{\mathcal{W}} = \langle\langle \tau_-(\mathcal{W}) \gamma^\alpha \rangle\rangle_0 \gamma_\alpha + \pi(\mathcal{W}). \quad (4)$$

An immediate consequence of this formula is the following:

$$T(\mathcal{W}) = \tau_+(\mathcal{W}) - \mathcal{S} \bullet \tau_+(\mathcal{W}) \quad (5)$$

for any vector field  $\mathcal{W}$ . Based on this formula and the fact that  $\mathcal{W} - \mathcal{S} \bullet \mathcal{W} = 0$  if  $\mathcal{W}$  is a primary vector field, we can interpret the operator  $T$  as a measurement of the difference between  $\mathcal{S}$  and a true identity of the quantum product on the big phase space. We have the following characterization for the vectors which are equivalent to 0.

**Lemma 1.4** For any vector field  $\mathcal{W}$ ,

$$\begin{aligned} \mathcal{W} \sim 0 &\iff \mathcal{W} = T(\tau_-(\mathcal{W})) \\ &\iff \mathcal{W} = T(\mathcal{U}) \text{ for some } \mathcal{U}. \end{aligned}$$

**Remark:** First derivatives of the string equation gives us  $\langle\langle \mathcal{S} \gamma^\alpha \rangle\rangle_0 = t_0^\alpha$  (cf. [LT, Lemma 1.1 (2)]). Therefore

$$T(\mathcal{S}) = \mathcal{D}$$

where  $\mathcal{D}$  is the *dilaton vector field* defined by  $\mathcal{D} = -\sum_{m,\alpha} \tilde{t}_m^\alpha \tau_m(\gamma_\alpha)$ . This lemma in particular implies that  $\mathcal{D} \sim 0$ , which also follows from the *dilaton equation*

$$\langle\langle \mathcal{D} \rangle\rangle_g = (2g - 2)F_g + \frac{1}{24} \chi(V) \delta_{g,1}$$

(cf. [LT, Lemma 1.2]). Moreover  $T(\mathcal{D}) \sim 0$  is equivalent to [LT, Lemma 5.2 (2)], which is also equivalent to the genus-0  $\tilde{\mathcal{L}}_1$  constraint. Similar reasoning also applies to the genus-0  $\tilde{\mathcal{L}}_2$  constraint by considering the vector field  $T(R(\mathcal{D}))$  where  $R$  is defined in Definition 3.8.

**Proof of Lemma 1.4:** By equation (5), for any vector field  $\mathcal{W}$ ,

$$\begin{aligned}\mathcal{W} &= \tau_+(\tau_-(\mathcal{W})) + \pi(\mathcal{W}) \\ &= T(\tau_-(\mathcal{W})) + \mathcal{S} \bullet \tau_+(\tau_-(\mathcal{W})) + \pi(\mathcal{W}) \\ &= T(\tau_-(\mathcal{W})) + \mathcal{S} \bullet (\mathcal{W} - \pi(\mathcal{W})) + \pi(\mathcal{W}).\end{aligned}$$

Since  $\mathcal{S} \bullet \pi(\mathcal{W}) = \pi(\mathcal{W})$ , we have

$$\mathcal{W} = T(\tau_-(\mathcal{W})) + \overline{\mathcal{W}}. \quad (6)$$

In particular,  $\mathcal{W} = T(\tau_-(\mathcal{W}))$  if  $\mathcal{W} \sim 0$ . On the other hand, equation (5) implies  $T(\mathcal{U}) \sim 0$  for any  $\mathcal{U}$  since  $\overline{\mathcal{W}} \sim \mathcal{W}$  for all  $\mathcal{W}$ .  $\square$

Let  $\nabla$  be the covariant derivative defined by

$$\nabla_{\mathcal{V}} \mathcal{W} = \sum_{m,\alpha} (\mathcal{V} f_{m,\alpha}) \tau_m(\gamma_\alpha)$$

for any vector fields  $\mathcal{V}$  and  $\mathcal{W} = \sum_{m,\alpha} f_{m,\alpha} \tau_m(\gamma_\alpha)$ . Then we have

$$\mathcal{V} \langle\langle \mathcal{W}_1 \mathcal{W}_2 \dots \mathcal{W}_k \rangle\rangle_g = \langle\langle \mathcal{V} \mathcal{W}_1 \dots \mathcal{W}_k \rangle\rangle_g + \sum_{j=1}^k \langle\langle \mathcal{W}_1 \dots (\nabla_{\mathcal{V}} \mathcal{W}_j) \dots \mathcal{W}_k \rangle\rangle_g \quad (7)$$

for any vector fields  $\mathcal{V}$  and  $\mathcal{W}_j$ . In particular, we have

$$\nabla_{\mathcal{V}}(\mathcal{W} \bullet \mathcal{U}) = \langle\langle \mathcal{V} \mathcal{W} \mathcal{U} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha + (\nabla_{\mathcal{V}} \mathcal{W}) \bullet \mathcal{U} + \mathcal{W} \bullet (\nabla_{\mathcal{V}} \mathcal{U}) \quad (8)$$

for any vector fields  $\mathcal{U}$ ,  $\mathcal{V}$  and  $\mathcal{W}$ . The following formulas will be useful in the study of topological recursion relations:

**Lemma 1.5** *For all vector fields  $\mathcal{V}$  and  $\mathcal{W}$ ,*

- (1)  $\nabla_{\mathcal{V}} \tau_+(\mathcal{W}) = \tau_+(\nabla_{\mathcal{V}} \mathcal{W})$ .
- (2)  $\nabla_{\mathcal{V}} \tau_-(\mathcal{W}) = \tau_-(\nabla_{\mathcal{V}} \mathcal{W})$ .
- (3)  $\nabla_{\mathcal{V}} T(\mathcal{W}) = T(\nabla_{\mathcal{V}} \mathcal{W}) - \mathcal{V} \bullet \mathcal{W}$ .

**Proof:** It is straightforward to check (1) and (2). Hence

$$\begin{aligned}\nabla_{\mathcal{V}} T(\mathcal{W}) &= \nabla_{\mathcal{V}} \tau_+(\mathcal{W}) - (\mathcal{V} \langle\langle \mathcal{W} \gamma^\alpha \rangle\rangle_0) \gamma_\alpha \\ &= \tau_+(\nabla_{\mathcal{V}} \mathcal{W}) - \langle\langle (\nabla_{\mathcal{V}} \mathcal{W}) \gamma^\alpha \rangle\rangle_0 \gamma_\alpha - \langle\langle \mathcal{V} \mathcal{W} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\ &= T(\nabla_{\mathcal{V}} \mathcal{W}) - \mathcal{V} \bullet \mathcal{W}.\end{aligned}$$

This proves (3).  $\square$

**Corollary 1.6** *For any vector fields  $\mathcal{W}_1$ ,  $\mathcal{W}_2$ , and  $\mathcal{V}$ ,*

$$\langle\langle \mathcal{W}_1 \mathcal{W}_2 T(\mathcal{V}) \gamma^\alpha \rangle\rangle_0 \gamma_\alpha = \mathcal{W}_1 \bullet \mathcal{W}_2 \bullet \mathcal{V}.$$

**Proof:** By equation (8),

$$\langle\langle \mathcal{W}_1 \mathcal{W}_2 T(\mathcal{V}) \gamma^\alpha \rangle\rangle_0 \gamma_\alpha = \nabla_{\mathcal{W}_1}(\mathcal{W}_2 \bullet T(\mathcal{V})) - (\nabla_{\mathcal{W}_1} \mathcal{W}_2) \bullet T(\mathcal{V}) - \mathcal{W}_2 \bullet (\nabla_{\mathcal{W}_1} T(\mathcal{V})).$$

The first two terms on the right hand side vanish since  $T(\mathcal{V}) \sim 0$ . The corollary then follows from Lemma 1.5.  $\square$

As a consequence of equation (8), we also have

**Corollary 1.7** *For any vector fields  $\mathcal{W}_1$ ,  $\mathcal{W}_2$ , and  $\mathcal{V}$ ,*

$$\nabla_{T(\mathcal{V})}(\mathcal{W}_1 \bullet \mathcal{W}_2) = (\nabla_{T(\mathcal{V})} \mathcal{W}_1) \bullet \mathcal{W}_2 + \mathcal{W}_1 \bullet (\nabla_{T(\mathcal{V})} \mathcal{W}_2) + \mathcal{W}_1 \bullet \mathcal{W}_2 \bullet \mathcal{V}.$$

Since  $T(\mathcal{V}) \sim 0$ , this corollary implies that  $\nabla_{T^2(\mathcal{V})}$  is a derivation of the quantum product for any vector field  $\mathcal{V}$ . Since covariant derivatives preserve the space of primary vector fields,  $\nabla_{T^2(\mathcal{V})}$  and the quantum product produce a holomorphic vertex algebra structure on this space (cf. [Ka]).

Taking derivatives of the equation in Corollary 1.6, we have

$$\begin{aligned} \langle\langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 T(\mathcal{V}) \gamma^\alpha \rangle\rangle_0 \gamma_\alpha &= \langle\langle \{\mathcal{V} \bullet \mathcal{W}_1\} \mathcal{W}_2 \mathcal{W}_3 \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\ &\quad + \langle\langle \{\mathcal{V} \bullet \mathcal{W}_2\} \mathcal{W}_1 \mathcal{W}_3 \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\ &\quad + \langle\langle \mathcal{V} \mathcal{W}_1 \mathcal{W}_2 \{\mathcal{W}_3 \bullet \gamma^\alpha\} \rangle\rangle_0 \gamma_\alpha \end{aligned} \quad (9)$$

and

$$\begin{aligned} \langle\langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \mathcal{W}_4 T(\mathcal{V}) \gamma^\alpha \rangle\rangle_0 \gamma_\alpha &= \sum_{i=1}^3 \langle\langle \{\mathcal{V} \bullet \mathcal{W}_i\} \mathcal{W}_1 \dots \widehat{\mathcal{W}_i} \dots \mathcal{W}_4 \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\ &\quad + \langle\langle \mathcal{V} \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \{\mathcal{W}_4 \bullet \gamma^\alpha\} \rangle\rangle_0 \gamma_\alpha \\ &\quad + \langle\langle \mathcal{V} \mathcal{W}_1 \mathcal{W}_2 \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \mathcal{W}_3 \mathcal{W}_4 \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\ &\quad + \langle\langle \mathcal{V} \mathcal{W}_1 \mathcal{W}_3 \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \mathcal{W}_2 \mathcal{W}_4 \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\ &\quad + \langle\langle \mathcal{V} \mathcal{W}_2 \mathcal{W}_3 \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \mathcal{W}_1 \mathcal{W}_4 \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \end{aligned} \quad (10)$$

for all vector fields. These formulas will be useful later when we study the genus-2 Virasoro conjecture. For  $k \geq 1$ , an induction on  $k$  shows that

$$\langle\langle T^k(\mathcal{W}) \mathcal{V}_1 \dots \mathcal{V}_k \gamma^\alpha \rangle\rangle_0 \gamma_\alpha = 0. \quad (11)$$

We collect some useful formulas for the string vector field in the following

**Lemma 1.8**

- (1)  $\nabla_{\mathcal{W}} \mathcal{S} = -\tau_-(\mathcal{W})$ .
- (2)  $\langle\langle \mathcal{S} \mathcal{V} \mathcal{W} \gamma^\alpha \rangle\rangle_0 = \langle\langle \tau_-(\mathcal{V}) \mathcal{W} \gamma^\alpha \rangle\rangle_0 + \langle\langle \mathcal{V} \tau_-(\mathcal{W}) \gamma^\alpha \rangle\rangle_0$ .
- (3)  $\mathcal{W} \langle\langle \mathcal{S} \mathcal{S} \gamma^\alpha \rangle\rangle_0 = \langle\langle \mathcal{W} \tau_-(\mathcal{S}) \gamma^\alpha \rangle\rangle_0 - \langle\langle \tau_-(\mathcal{W}) \mathcal{S} \gamma^\alpha \rangle\rangle_0$ .



**Proof:** The first formula follows from the definition of  $\mathcal{S}$ . The genus-0 string equation implies that

$$\langle\langle \mathcal{S} \tau_m(\gamma_\mu) \tau_n(\gamma_\nu) \gamma^\alpha \rangle\rangle_0 = \langle\langle \tau_{m-1}(\gamma_\mu) \tau_n(\gamma_\nu) \gamma^\alpha \rangle\rangle_0 + \langle\langle \tau_m(\gamma_\mu) \tau_{n-1}(\gamma_\nu) \gamma^\alpha \rangle\rangle_0$$

for all  $\tau_m(\gamma_\mu)$  and  $\tau_n(\gamma_\nu)$ . By linearity of the correlation functions, this implies the second formula. The last formula follows from the first two formulas and equation (7).  $\square$

Lemma 1.8 (2) and equation (8) implies  $\nabla_{\mathcal{S}}$  is a derivation of the quantum product on the space of primary vector fields and therefore also produce a holomorphic vertex algebra structure.

Using Lemma 1.8 (1) and Lemma 1.5 (3), we obtain

$$\nabla_{T^m(\mathcal{S})} T^k(\mathcal{S}) = -T^{k+m-1}(\mathcal{S}).$$

Therefore,

$$[T^k(\mathcal{S}), T^m(\mathcal{S})] = 0$$

for all  $k, m \geq 0$ .

Another application of Lemma 1.8 (1) is that

$$\langle\langle \mathcal{S} \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g = \sum_{i=1}^k \langle\langle \mathcal{W}_1 \cdots \{\tau_-(\mathcal{W}_i)\} \cdots \mathcal{W}_k \rangle\rangle_g + \delta_{g,0} \nabla_{\mathcal{W}_1, \dots, \mathcal{W}_k}^k \left( \frac{1}{2} \eta_{\alpha\beta} t_0^\alpha t_0^\beta \right) \quad (12)$$

for any vector fields  $\mathcal{W}_1, \dots, \mathcal{W}_k$ , where  $\nabla^k$  is the  $k$ -th covariant derivative. This formula is obtained by repeatedly taking derivatives on both sides of the string equation and applying equation (7).

## 2 Applications to topological recursion relations

It seems that the quantum product on the big phase space provides appropriate algebraic machinery for studying problems on the big phase space, e.g. the Virasoro conjecture and topological recursion relations. We first consider the topological recursion relations in this section and postpone the Virasoro conjecture to later sections.

Lemma 1.4 indicates that the transformation  $T$  trivializes vector fields at the genus-0 level. In some sense, all known topological recursion relations seem to indicate how this operator trivialize vector fields for various genera. Let us first see the cases of genus less than or equal to 2.

Since

$$T(\tau_n(\gamma_\alpha)) = \tau_{n+1}(\gamma_\alpha) - \langle\langle \tau_n(\gamma_\alpha) \gamma^\mu \rangle\rangle_0 \gamma_\mu,$$

the coefficient of  $\gamma_\nu$  for  $T(\tau_n(\gamma_\alpha)) \bullet \tau_m(\gamma_\beta)$  is

$$\langle\langle \tau_{n+1}(\gamma_\alpha) \tau_m(\gamma_\beta) \gamma^\nu \rangle\rangle_0 - \langle\langle \tau_n(\gamma_\alpha) \gamma^\mu \rangle\rangle_0 \langle\langle \gamma_\mu \tau_m(\gamma_\beta) \gamma^\nu \rangle\rangle_0.$$

The vanishing of this quantity is the most important case of the genus-0 topological recursion relation (in particular, it implies the associativity of the quantum product on

the big phase space). Therefore  $T(\tau_n(\gamma_\alpha)) \sim 0$  is equivalent to this special case of the genus-0 topological recursion relation.

The genus-1 topological recursion relation is the following:

$$\langle\langle \tau_{n+1}(\gamma_\alpha) \rangle\rangle_1 = \langle\langle \tau_n(\gamma_\alpha) \gamma^\mu \rangle\rangle_0 \langle\langle \gamma_\mu \rangle\rangle_1 + \frac{1}{24} \langle\langle \tau_n(\gamma_\alpha) \gamma^\mu \gamma_\mu \rangle\rangle_0.$$

This formula is equivalent to

$$\langle\langle T(\mathcal{W}) \rangle\rangle_1 = \frac{1}{24} \langle\langle \mathcal{W} \gamma^\mu \gamma_\mu \rangle\rangle_0 \quad (13)$$

for any vector field  $\mathcal{W}$ . For  $g > 0$ , we call a vector field  $\mathcal{W}$  *trivial at the genus- $g$  level* if  $\langle\langle \mathcal{W} \rangle\rangle_g$  can be represented by data of genera less than  $g$ . Then the genus-1 topological recursion relation just means that  $T(\mathcal{W})$  is trivial at the genus-1 level for all  $\mathcal{W}$ .

Taking derivatives of equation (13) and using Lemma 1.5, we have

$$\langle\langle T(\mathcal{W}) \mathcal{V} \rangle\rangle_1 = \langle\langle \{\mathcal{W} \bullet \mathcal{V}\} \rangle\rangle_1 + \frac{1}{24} \langle\langle \mathcal{W} \mathcal{V} \gamma^\mu \gamma_\mu \rangle\rangle_0 \quad (14)$$

and

$$\begin{aligned} \langle\langle T(\mathcal{W}) \mathcal{V}_1 \mathcal{V}_2 \rangle\rangle_1 &= \langle\langle \{\mathcal{W} \bullet \mathcal{V}_1\} \mathcal{V}_2 \rangle\rangle_1 + \langle\langle \{\mathcal{W} \bullet \mathcal{V}_2\} \mathcal{V}_1 \rangle\rangle_1 + \langle\langle \mathcal{W} \mathcal{V}_1 \mathcal{V}_2 \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_1 \\ &\quad + \frac{1}{24} \langle\langle \mathcal{W} \mathcal{V}_1 \mathcal{V}_2 \gamma^\mu \gamma_\mu \rangle\rangle_0 \end{aligned} \quad (15)$$

for all vector fields. These formulas will be used later.

The genus-2 topological recursion relations are much more complicated than genus-0 and genus-1 topological recursion relations. The following two recursion relations were found in [G1]:

$$\begin{aligned} \langle\langle \tau_{i+2}(x) \rangle\rangle_2 &= \langle\langle \tau_{i+1}(x) \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_2 + \langle\langle \tau_i(x) \gamma^\alpha \rangle\rangle_0 \langle\langle \tau_1(\gamma_\alpha) \rangle\rangle_2 \\ &\quad - \langle\langle \tau_i(x) \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_2 + \frac{7}{10} \langle\langle \tau_i(x) \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma_\beta \rangle\rangle_1 \\ &\quad + \frac{1}{10} \langle\langle \tau_i(x) \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_1 - \frac{1}{240} \langle\langle \tau_i(x) \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma^\alpha \gamma_\beta \gamma^\beta \rangle\rangle_0 \\ &\quad + \frac{13}{240} \langle\langle \tau_i(x) \gamma_\alpha \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 + \frac{1}{960} \langle\langle \tau_i(x) \gamma^\alpha \gamma_\alpha \gamma^\beta \gamma_\beta \rangle\rangle_0 \end{aligned} \quad (16)$$

for  $i \geq 0$  and  $x \in \{\gamma_1, \dots, \gamma_N\}$ , and

$$\begin{aligned} \langle\langle \tau_{i+1}(x) \tau_{j+1}(y) \rangle\rangle_2 &= \langle\langle \tau_{i+1}(x) \gamma^\alpha \rangle\rangle_2 \langle\langle \gamma_\alpha \tau_j(y) \rangle\rangle_0 + \langle\langle \tau_i(x) \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \tau_{j+1}(y) \rangle\rangle_2 \\ &\quad - \langle\langle \tau_i(x) \gamma^\alpha \rangle\rangle_0 \langle\langle \tau_j(y) \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_2 + 3 \langle\langle \tau_i(x) \tau_j(y) \gamma^\alpha \rangle\rangle_0 \langle\langle \tau_1(\gamma_\alpha) \rangle\rangle_2 \\ &\quad - 3 \langle\langle \tau_i(x) \tau_j(y) \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_2 + \frac{13}{10} \langle\langle \tau_i(x) \tau_j(y) \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma_\beta \rangle\rangle_1 \\ &\quad + \frac{4}{5} \langle\langle \tau_i(x) \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma^\alpha \tau_j(y) \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 + \frac{4}{5} \langle\langle \tau_i(x) \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \tau_j(y) \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma_\beta \rangle\rangle_1 \end{aligned}$$

$$\begin{aligned}
& -\frac{4}{5} \langle\langle \tau_i(x) \tau_j(y) \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma^\beta \rangle\rangle_1 \langle\langle \gamma_\beta \rangle\rangle_1 + \frac{23}{240} \langle\langle \tau_i(x) \tau_j(y) \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 \\
& + \frac{1}{48} \langle\langle \tau_i(x) \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \tau_j(y) \rangle\rangle_1 + \frac{1}{48} \langle\langle \tau_j(y) \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \tau_i(x) \rangle\rangle_1 \\
& - \frac{1}{80} \langle\langle \tau_i(x) \tau_j(y) \gamma^\alpha \rangle\rangle_1 \langle\langle \gamma_\alpha \gamma^\beta \gamma_\beta \rangle\rangle_0 + \frac{7}{30} \langle\langle \tau_i(x) \tau_j(y) \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_1 \\
& + \frac{1}{30} \langle\langle \tau_i(x) \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \tau_j(y) \rangle\rangle_1 + \frac{1}{30} \langle\langle \tau_j(y) \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \tau_i(x) \rangle\rangle_1 \\
& - \frac{1}{30} \langle\langle \tau_i(x) \tau_j(y) \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma^\beta \gamma_\beta \rangle\rangle_1 + \frac{1}{576} \langle\langle \tau_i(x) \tau_j(y) \gamma^\alpha \gamma_\alpha \gamma^\beta \gamma_\beta \rangle\rangle_0 \quad (17)
\end{aligned}$$

for  $i, j \geq 0$  and  $x, y \in \{\gamma_1, \dots, \gamma_N\}$ . Another genus-2 recursion relation was found in [BP]:

$$\begin{aligned}
0 = & \sum_{\sigma \in S_3} -2 \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma^\alpha \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_2 \\
& + 2 \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \gamma^\alpha \rangle\rangle_0 \langle\langle \tau_1(\gamma_\alpha) \rangle\rangle_2 \\
& - 2 \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_2 \\
& + 3 \langle\langle \tau_{1+i_{\sigma(1)}}(x_{\sigma(1)}) \gamma_\alpha \rangle\rangle_2 \langle\langle \gamma^\alpha \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \rangle\rangle_0 \\
& - 3 \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \gamma^\alpha \rangle\rangle_0 \langle\langle \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_2 \\
& - 3 \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_1(\gamma_\alpha) \rangle\rangle_2 \langle\langle \gamma^\alpha \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \rangle\rangle_0 \\
& + 3 \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \gamma_\alpha \rangle\rangle_2 \langle\langle \gamma^\alpha \gamma_\beta \rangle\rangle_0 \langle\langle \gamma^\beta \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \rangle\rangle_0 \\
& + \frac{1}{5} \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma_\beta \rangle\rangle_1 \\
& - \frac{6}{5} \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma_\beta \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \rangle\rangle_1 \\
& + \frac{12}{5} \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma_\beta \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \rangle\rangle_1 \\
& - \frac{18}{5} \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma^\alpha \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \rangle\rangle_1 \\
& - \frac{6}{5} \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma^\beta \rangle\rangle_1 \langle\langle \gamma_\beta \rangle\rangle_1 \\
& + \frac{9}{5} \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma^\alpha \gamma_\beta \rangle\rangle_1 \langle\langle \gamma^\beta \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \rangle\rangle_0 \\
& - \frac{6}{5} \langle\langle \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma^\alpha \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \gamma_\beta \rangle\rangle_1 \langle\langle \gamma^\beta \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \rangle\rangle_0 \\
& + \frac{1}{120} \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 \\
& - \frac{3}{40} \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \rangle\rangle_1 \\
& + \frac{3}{40} \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \rangle\rangle_1 \\
& - \frac{1}{120} \langle\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma^\alpha \gamma^\beta \gamma_\beta \rangle\rangle_0
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{10} \left\langle\left\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \gamma^\alpha \gamma^\beta \right\rangle\right\rangle_0 \left\langle\left\langle \gamma_\alpha \gamma_\beta \right\rangle\right\rangle_1 \\
& - \frac{3}{10} \left\langle\left\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \gamma_\alpha \gamma_\beta \right\rangle\right\rangle_1 \left\langle\left\langle \gamma^\alpha \gamma^\beta \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \right\rangle\right\rangle_0 \\
& + \frac{1}{10} \left\langle\left\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \gamma_\alpha \gamma_\beta \right\rangle\right\rangle_1 \left\langle\left\langle \gamma^\alpha \gamma^\beta \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \right\rangle\right\rangle_0 \\
& - \frac{1}{20} \left\langle\left\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \gamma^\alpha \right\rangle\right\rangle_0 \left\langle\left\langle \gamma^\alpha \gamma^\beta \gamma_\beta \right\rangle\right\rangle_1 \\
& - \frac{1}{20} \left\langle\left\langle \tau_{i_{\sigma(1)}}(x_{\sigma(1)}) \gamma^\alpha \gamma_\alpha \gamma_\beta \right\rangle\right\rangle_1 \left\langle\left\langle \gamma^\beta \tau_{i_{\sigma(2)}}(x_{\sigma(2)}) \tau_{i_{\sigma(3)}}(x_{\sigma(3)}) \right\rangle\right\rangle_0, \tag{18}
\end{aligned}$$

where  $i_1, i_2, i_3 \geq 0$ ,  $x_1, x_2, x_3 \in \{\gamma_1, \dots, \gamma_N\}$ , and  $S_3$  is the symmetry group of three elements.

Unlike the genus-1 case,  $T(\mathcal{W})$  is no longer trivial at the genus-2 level in general. In fact, the genus-2 dilaton equation implies

$$\left\langle\left\langle T(\mathcal{S}) \right\rangle\right\rangle_2 = \left\langle\left\langle \mathcal{D} \right\rangle\right\rangle_2 = 2F_2.$$

Unless that  $F_2$  can be expressed as a function of  $F_0$  and  $F_1$ ,  $T(\mathcal{S})$  is not trivial at the genus-2 level. However, the topological recursion relation (16) implies that  $T^2(\mathcal{W}) := T(T(\mathcal{W}))$  is trivial at the genus-2 level for all vector field  $\mathcal{W}$ .

The genus-2 topological recursion relations (16) - (18) can be represented in the following forms respectively: For any vector fields  $\mathcal{W}$ ,  $\mathcal{V}$ ,  $\mathcal{W}_1$ ,  $\mathcal{W}_2$ , and  $\mathcal{W}_3$  on the big phase space,

$$\left\langle\left\langle T^2(\mathcal{W}) \right\rangle\right\rangle_2 = A_1(\mathcal{W}), \tag{19}$$

$$\left\langle\left\langle T(\mathcal{W}) T(\mathcal{V}) \right\rangle\right\rangle_2 - 3 \left\langle\left\langle T(\mathcal{W} \bullet \mathcal{V}) \right\rangle\right\rangle_2 = A_2(\mathcal{W}, \mathcal{V}), \tag{20}$$

and

$$\begin{aligned}
& 2 \left\langle\left\langle \{\mathcal{W}_1 \bullet \mathcal{W}_2 \bullet \mathcal{W}_3\} \right\rangle\right\rangle_2 - 2 \left\langle\left\langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \gamma^\alpha \right\rangle\right\rangle_0 \left\langle\left\langle T(\gamma_\alpha) \right\rangle\right\rangle_2 \\
& - \left\langle\left\langle T(\mathcal{W}_1) \{\mathcal{W}_2 \bullet \mathcal{W}_3\} \right\rangle\right\rangle_2 + \left\langle\left\langle \mathcal{W}_1 T(\mathcal{W}_2 \bullet \mathcal{W}_3) \right\rangle\right\rangle_2 \\
& - \left\langle\left\langle T(\mathcal{W}_2) \{\mathcal{W}_1 \bullet \mathcal{W}_3\} \right\rangle\right\rangle_2 + \left\langle\left\langle \mathcal{W}_2 T(\mathcal{W}_1 \bullet \mathcal{W}_3) \right\rangle\right\rangle_2 \\
& - \left\langle\left\langle T(\mathcal{W}_3) \{\mathcal{W}_1 \bullet \mathcal{W}_2\} \right\rangle\right\rangle_2 + \left\langle\left\langle \mathcal{W}_3 T(\mathcal{W}_1 \bullet \mathcal{W}_2) \right\rangle\right\rangle_2 = B(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3), \tag{21}
\end{aligned}$$

where

$$\begin{aligned}
A_1(\mathcal{W}) &= \frac{7}{10} \left\langle\left\langle \gamma_\alpha \right\rangle\right\rangle_1 \left\langle\left\langle \{\gamma^\alpha \bullet \mathcal{W}\} \right\rangle\right\rangle_1 + \frac{1}{10} \left\langle\left\langle \gamma_\alpha \{\gamma^\alpha \bullet \mathcal{W}\} \right\rangle\right\rangle_1 \\
& - \frac{1}{240} \left\langle\left\langle \mathcal{W} \{\gamma_\alpha \bullet \gamma^\alpha\} \right\rangle\right\rangle_1 + \frac{13}{240} \left\langle\left\langle \mathcal{W} \gamma_\alpha \gamma^\alpha \gamma^\beta \right\rangle\right\rangle_0 \left\langle\left\langle \gamma_\beta \right\rangle\right\rangle_1 \\
& + \frac{1}{960} \left\langle\left\langle \mathcal{W} \gamma^\alpha \gamma_\alpha \gamma^\beta \gamma_\beta \right\rangle\right\rangle_0, \\
A_2(\mathcal{W}, \mathcal{V}) &= \frac{13}{10} \left\langle\left\langle \mathcal{W} \mathcal{V} \gamma^\alpha \gamma^\beta \right\rangle\right\rangle_0 \left\langle\left\langle \gamma_\alpha \right\rangle\right\rangle_1 \left\langle\left\langle \gamma_\beta \right\rangle\right\rangle_1 + \frac{4}{5} \left\langle\left\langle \mathcal{W} \gamma^\alpha \right\rangle\right\rangle_1 \left\langle\left\langle \{\gamma_\alpha \bullet \mathcal{V}\} \right\rangle\right\rangle_1
\end{aligned}$$

$$\begin{aligned}
& +\frac{4}{5} \langle\langle \mathcal{V} \gamma^\alpha \rangle\rangle_1 \langle\langle \{\gamma_\alpha \bullet \mathcal{W}\} \rangle\rangle_1 - \frac{4}{5} \langle\langle \{\mathcal{W} \bullet \mathcal{V}\} \gamma^\alpha \rangle\rangle_1 \langle\langle \gamma_\alpha \rangle\rangle_1 \\
& +\frac{23}{240} \langle\langle \mathcal{W} \mathcal{V} \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 + \frac{1}{48} \langle\langle \mathcal{W} \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \mathcal{V} \rangle\rangle_1 \\
& +\frac{1}{48} \langle\langle \mathcal{V} \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \mathcal{W} \rangle\rangle_1 - \frac{1}{80} \langle\langle \mathcal{W} \mathcal{V} \{\gamma^\alpha \bullet \gamma_\alpha\} \rangle\rangle_1 \\
& +\frac{7}{30} \langle\langle \mathcal{W} \mathcal{V} \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_1 + \frac{1}{30} \langle\langle \gamma_\alpha \{\gamma^\alpha \bullet \mathcal{W}\} \mathcal{V} \rangle\rangle_1 \\
& +\frac{1}{30} \langle\langle \gamma_\alpha \{\gamma^\alpha \bullet \mathcal{V}\} \mathcal{W} \rangle\rangle_1 - \frac{1}{30} \langle\langle \{\mathcal{W} \bullet \mathcal{V}\} \gamma_\alpha \gamma^\alpha \rangle\rangle_1 \\
& +\frac{1}{576} \langle\langle \mathcal{W} \mathcal{V} \gamma^\alpha \gamma_\alpha \gamma^\beta \gamma_\beta \rangle\rangle_0,
\end{aligned}$$

and

$$\begin{aligned}
& B(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) \\
= & \frac{1}{5} \langle\langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma_\beta \rangle\rangle_1 - \frac{6}{5} \langle\langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma^\beta \rangle\rangle_1 \langle\langle \gamma_\beta \rangle\rangle_1 \\
& +\frac{1}{120} \langle\langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 - \frac{1}{120} \langle\langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \{\gamma^\alpha \bullet \gamma_\alpha\} \rangle\rangle_1 \\
& +\frac{1}{10} \langle\langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_1 - \frac{1}{20} \langle\langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma^\beta \gamma_\beta \rangle\rangle_1 \\
& -\frac{1}{5} \sum_{\sigma \in S_3} \langle\langle \mathcal{W}_{\sigma(1)} \mathcal{W}_{\sigma(2)} \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma_\beta \mathcal{W}_{\sigma(3)} \rangle\rangle_1 \\
& +\frac{2}{5} \sum_{\sigma \in S_3} \langle\langle \{\mathcal{W}_{\sigma(1)} \bullet \gamma_\alpha\} \rangle\rangle_1 \langle\langle \gamma^\alpha \mathcal{W}_{\sigma(2)} \mathcal{W}_{\sigma(3)} \rangle\rangle_1 \\
& -\frac{3}{5} \sum_{\sigma \in S_3} \langle\langle \mathcal{W}_{\sigma(1)} \{\mathcal{W}_{\sigma(2)} \bullet \gamma^\alpha\} \rangle\rangle_1 \langle\langle \gamma_\alpha \mathcal{W}_{\sigma(3)} \rangle\rangle_1 \\
& +\frac{3}{10} \sum_{\sigma \in S_3} \langle\langle \mathcal{W}_{\sigma(1)} \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma^\alpha \{\mathcal{W}_{\sigma(2)} \bullet \mathcal{W}_{\sigma(3)}\} \rangle\rangle_1 \\
& -\frac{1}{5} \sum_{\sigma \in S_3} \langle\langle \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma^\alpha \mathcal{W}_{\sigma(1)} \{\mathcal{W}_{\sigma(2)} \bullet \mathcal{W}_{\sigma(3)}\} \rangle\rangle_1 \\
& -\frac{1}{80} \sum_{\sigma \in S_3} \langle\langle \mathcal{W}_{\sigma(1)} \mathcal{W}_{\sigma(2)} \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \mathcal{W}_{\sigma(3)} \rangle\rangle_1 \\
& +\frac{1}{80} \sum_{\sigma \in S_3} \langle\langle \mathcal{W}_{\sigma(1)} \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \mathcal{W}_{\sigma(2)} \mathcal{W}_{\sigma(3)} \rangle\rangle_1 \\
& -\frac{1}{20} \sum_{\sigma \in S_3} \langle\langle \mathcal{W}_{\sigma(1)} \gamma_\alpha \gamma_\beta \rangle\rangle_1 \langle\langle \gamma^\alpha \gamma^\beta \mathcal{W}_{\sigma(2)} \mathcal{W}_{\sigma(3)} \rangle\rangle_0 \\
& +\frac{1}{60} \sum_{\sigma \in S_3} \langle\langle \mathcal{W}_{\sigma(1)} \mathcal{W}_{\sigma(2)} \gamma_\alpha \{\gamma^\alpha \bullet \mathcal{W}_{\sigma(3)}\} \rangle\rangle_1 \\
& -\frac{1}{120} \sum_{\sigma \in S_3} \langle\langle \mathcal{W}_{\sigma(1)} \gamma^\alpha \gamma_\alpha \{\mathcal{W}_{\sigma(2)} \bullet \mathcal{W}_{\sigma(3)}\} \rangle\rangle_1.
\end{aligned}$$

In particular,  $A_1$ ,  $A_2$ , and  $B$  are symmetric tensors only depending on genus-0 and genus-1 data.

We now study relations among these genus-2 recursion relations. We first observe that equation (19) is equivalent to

$$\langle\langle T(\mathcal{W}) \rangle\rangle_2 = \langle\langle T(\overline{\mathcal{W}}) \rangle\rangle_2 + A_1(\tau_-(\mathcal{W})). \quad (22)$$

for any vector field  $\mathcal{W}$ . The following two lemmas are formal consequences of (19) or its equivalent form (22).

**Lemma 2.1** *Equation (19) implies*

$$\langle\langle T^2(\mathcal{V}) \mathcal{W} \rangle\rangle_2 = \langle\langle T(\mathcal{V} \bullet \mathcal{W}) \rangle\rangle_2 + (\nabla_{\mathcal{W}} A_1)(\mathcal{V})$$

for any vector fields  $\mathcal{V}$  and  $\mathcal{W}$ , where  $\nabla_{\mathcal{W}} A_1$  is the covariant derivative of  $A_1$  defined by

$$(\nabla_{\mathcal{W}} A_1)(\mathcal{V}) = \mathcal{W} A_1(\mathcal{V}) - A_1(\nabla_{\mathcal{W}} \mathcal{V}).$$

**Proof:** By equation (7),

$$\langle\langle T^2(\mathcal{V}) \mathcal{W} \rangle\rangle_2 = \mathcal{W} \langle\langle T^2(\mathcal{V}) \rangle\rangle_2 - \langle\langle \{ \nabla_{\mathcal{W}} T^2(\mathcal{V}) \} \rangle\rangle_2.$$

By equation (19) and Lemma 1.5, we have

$$\langle\langle T^2(\mathcal{V}) \mathcal{W} \rangle\rangle_2 = \mathcal{W} A_1(\mathcal{V}) - \langle\langle T(\nabla_{\mathcal{W}} T(\mathcal{V})) \rangle\rangle_2 + \langle\langle \{ \mathcal{W} \bullet T(\mathcal{V}) \} \rangle\rangle_2.$$

The last term on the right hand side vanishes since  $T(\mathcal{V}) \sim 0$ . Using Lemma 1.5 again, we have

$$\langle\langle T^2(\mathcal{V}) \mathcal{W} \rangle\rangle_2 = \mathcal{W} A_1(\mathcal{V}) - \langle\langle T^2(\nabla_{\mathcal{W}} \mathcal{V}) \rangle\rangle_2 + \langle\langle T(\mathcal{W} \bullet \mathcal{V}) \rangle\rangle_2.$$

The lemma then follows from equation (19).  $\square$

**Corollary 2.2** *Equation (19) implies*

$$\langle\langle T(\mathcal{V}) T(\overline{\mathcal{W}}) \rangle\rangle_2 = \langle\langle T(\mathcal{V}) T(\mathcal{W}) \rangle\rangle_2 - (\nabla_{T(\mathcal{V})} A_1)(\tau_-(\mathcal{W}))$$

for any vector field  $\mathcal{V}$  and  $\mathcal{W}$ .

**Proof:** Since  $\mathcal{W} - \overline{\mathcal{W}} = T(\tau_-(\mathcal{W}))$ ,

$$\langle\langle T(\mathcal{V}) T(\mathcal{W}) \rangle\rangle_2 - \langle\langle T(\mathcal{V}) T(\overline{\mathcal{W}}) \rangle\rangle_2 = \langle\langle T(\mathcal{V}) T^2(\tau_-(\mathcal{W})) \rangle\rangle_2.$$

Applying Lemma 2.1 and using the fact  $T(\mathcal{V}) \sim 0$ , we obtain the desired formula.  $\square$

Since  $\overline{T(\mathcal{W})} = 0$ , if  $\mathcal{W}$  is replaced by  $T(\mathcal{W})$  in Corollary 2.2, we obtain

$$\langle\langle T(\mathcal{V}) T^2(\mathcal{W}) \rangle\rangle_2 = (\nabla_{T(\mathcal{V})} A_1)(\mathcal{W})$$

for any vector field  $\mathcal{V}$  and  $\mathcal{W}$ . On the other hand, since  $T(\mathcal{V}) \sim 0$ , if  $\mathcal{W}$  is replaced by  $T(\mathcal{W})$  in equation (20), we obtain

$$\langle\langle T(\mathcal{V}) T^2(\mathcal{W}) \rangle\rangle_2 = A_2(\mathcal{V}, T(\mathcal{W})).$$

There are two consequences from these equations:

**Corollary 2.3** *Equations (19) and (20) imply*

$$A_2(\mathcal{V}, T(\mathcal{W})) = \left( \nabla_{T(\mathcal{V})} A_1 \right) (\mathcal{W})$$

for any vector fields  $\mathcal{V}$  and  $\mathcal{W}$ .

**Remark:** Note that both sides of the equation in Corollary 2.3 only involve genus-0 and genus-1 data. It might be possible to prove this relation by known genus-1 relations. Since our main interest in this paper is to study higher genus cases, we will not do this here.

**Corollary 2.4** *Modulo the genus-1 relation in Corollary 2.3, equation (20) follows from equation (19) if one of the vector fields is equivalent to 0.*

A special case of Corollary 2.3 is the following

**Corollary 2.5**

$$A_2(\mathcal{S}, \mathcal{W}) = 3A_1(\tau_-(\mathcal{W}))$$

for any vector field  $\mathcal{W}$ .

**Proof:** This formula can be proved directly by using the derivatives of the string equation (12). On the other hand, it follows from derivatives of the string equation and the dilaton equation that

$$A_2(\mathcal{S}, \mathcal{W}) = 0$$

if  $\mathcal{W}$  is a primary vector field and

$$\nabla_{\mathcal{D}} A_1 = 3 A_1.$$

Therefore the formula in this corollary also follows from Corollary 2.3 by using equation (6).  $\square$

In the opposite direction of Corollary 2.4, we have the following

**Theorem 2.6** *The topological recursion relation (19) is a formal consequence of (20).*

**Proof:** The first derivatives of the genus-2 string equation and dilaton equation have the following form

$$\langle\langle \mathcal{S} \mathcal{W} \rangle\rangle_2 = \langle\langle \tau_-(\mathcal{W}) \rangle\rangle_2, \quad \langle\langle \mathcal{D} \mathcal{W} \rangle\rangle_2 = 3 \langle\langle \mathcal{W} \rangle\rangle_2 \quad (23)$$

for any vector field  $\mathcal{W}$ . Since  $T(\mathcal{S}) = \mathcal{D}$ , applying equation (20) for  $\mathcal{V} = \mathcal{S}$ , we obtain

$$\begin{aligned} A_2(\mathcal{S}, \mathcal{W}) &= \langle\langle T(\mathcal{W}) \mathcal{D} \rangle\rangle_2 - 3 \langle\langle T(\overline{\mathcal{W}}) \rangle\rangle_2 \\ &= 3 \langle\langle T(\mathcal{W}) \rangle\rangle_2 - 3 \langle\langle T(\overline{\mathcal{W}}) \rangle\rangle_2 \\ &= 3 \langle\langle T(\mathcal{W} - \overline{\mathcal{W}}) \rangle\rangle_2 = 3 \langle\langle T^2(\tau_-(\mathcal{W})) \rangle\rangle_2 \end{aligned}$$

for any vector field  $\mathcal{W}$ . This is equivalent to equation (19) because of the Corollary 2.5.  
 $\square$

Next, we study the relations between equations (20) and (21). If we replace  $\mathcal{W}_3$  by  $T(\mathcal{V})$  in equation (21), we obtain

$$\begin{aligned} & -2 \langle\langle \mathcal{W}_1 \mathcal{W}_2 T(\mathcal{V}) \gamma^\alpha \rangle\rangle_0 \langle\langle T(\gamma_\alpha) \rangle\rangle_2 - \langle\langle T^2(\mathcal{V}) \{\mathcal{W}_1 \bullet \mathcal{W}_2\} \rangle\rangle_2 + \langle\langle T(\mathcal{V}) T(\mathcal{W}_1 \bullet \mathcal{W}_2) \rangle\rangle_2 \\ & = B(\mathcal{W}_1, \mathcal{W}_2, T(\mathcal{V})). \end{aligned}$$

By Corollary 1.6, this is equivalent to

$$\begin{aligned} & -2 \langle\langle T(\mathcal{W}_1 \bullet \mathcal{W}_2 \bullet \mathcal{V}) \rangle\rangle_2 - \langle\langle T^2(\mathcal{V}) \{\mathcal{W}_1 \bullet \mathcal{W}_2\} \rangle\rangle_2 + \langle\langle T(\mathcal{V}) T(\mathcal{W}_1 \bullet \mathcal{W}_2) \rangle\rangle_2 \\ & = B(\mathcal{W}_1, \mathcal{W}_2, T(\mathcal{V})). \end{aligned}$$

On the other hand, by Theorem 2.6 and Lemma 2.1, we know that equation (20) implies

$$\begin{aligned} & -2 \langle\langle T(\mathcal{W}_1 \bullet \mathcal{W}_2 \bullet \mathcal{V}) \rangle\rangle_2 - \langle\langle T^2(\mathcal{V}) \{\mathcal{W}_1 \bullet \mathcal{W}_2\} \rangle\rangle_2 + \langle\langle T(\mathcal{V}) T(\mathcal{W}_1 \bullet \mathcal{W}_2) \rangle\rangle_2 \\ & = -(\nabla_{\mathcal{W}_1 \bullet \mathcal{W}_2} A_1)(\mathcal{V}) + A_2(\mathcal{W}_1 \bullet \mathcal{W}_2, \mathcal{V}). \end{aligned}$$

Therefore we have the following two consequences

**Lemma 2.7** *Equations (20) and (21) imply*

$$B(\mathcal{W}_1, \mathcal{W}_2, T(\mathcal{V})) = A_2(\mathcal{W}_1 \bullet \mathcal{W}_2, \mathcal{V}) - (\nabla_{\mathcal{W}_1 \bullet \mathcal{W}_2} A_1)(\mathcal{V})$$

*for all vector fields.*

**Remark:** Both sides of the equation in this lemma only involve genus-0 and genus-1 data. It might be possible to prove this equation directly using known genus-1 relation. Again we will not do this in this paper.

**Lemma 2.8** *Modulo the genus-1 relation in Lemma 2.7, equation (21) follows from equation (20) if one of the vector fields is equivalent to 0.*

In the opposite direction of Lemma 2.8, we have

**Theorem 2.9** *Modulo the genus-1 relation in Lemma 2.7, the topological recursion relation (20) is a formal consequence of the topological recursion relation (19) and (21).*

**Proof:** Since  $T(\mathcal{V}) \sim 0$ , applying equation (21) for  $\mathcal{W}_1 = \mathcal{S}$ ,  $\mathcal{W}_2 = T(\mathcal{V})$  and  $\mathcal{W}_3 = \mathcal{W}$ , we obtain

$$-2 \langle\langle \mathcal{S} T(\mathcal{V}) \mathcal{W} \gamma^\alpha \rangle\rangle_0 \langle\langle T(\gamma_\alpha) \rangle\rangle_2 - \langle\langle T^2(\mathcal{V}) \overline{\mathcal{W}} \rangle\rangle_2 + \langle\langle T(\mathcal{V}) T(\overline{\mathcal{W}}) \rangle\rangle_2 = B(\mathcal{S}, \mathcal{W}, T(\mathcal{V})).$$

By Lemma 1.8, we have

$$\langle\langle \mathcal{S} T(\mathcal{V}) \mathcal{W} \gamma^\alpha \rangle\rangle_0 \langle\langle T(\gamma_\alpha) \rangle\rangle_2 = \langle\langle T(\mathcal{V} \bullet \mathcal{W}) \rangle\rangle_2.$$



The theorem then follows from Lemma 2.1 and Corollary 2.2 (which in turn follows from equation (19)), and Lemma 1.1.  $\square$

We now make a remark on higher genus topological recursion relation. For genus  $g = 1, 2$ , the topological recursion relations (13) and (19) are derived by using a formula for expressing the tautological class  $\psi_1^g$  (i.e. the  $g$ -th power of the first Chern class of the tautological line bundle defined by the cotangent space of each curve at the marked point) on the moduli space of stable curves  $\overline{\mathcal{M}}_{g,1}$  in terms of boundary classes. For general  $g$ , it was conjectured in [G2] that polynomials of degree  $g$  in the tautological classes  $\psi_i$  are boundary classes on  $\overline{\mathcal{M}}_{g,n}$ . This conjecture was proved in [Io]. A somewhat stronger version for a special case of this conjecture would be  $\psi_1^g$  is equal to a boundary class in  $\overline{\mathcal{M}}_{g,1}$  without genus- $g$  components. This would imply the following

$$T^g(\mathcal{W}) \text{ is trivial at the genus-}g \text{ level for } g \geq 1. \quad (24)$$

The following topological recursion relation for all  $g \geq 1$  was derived in [EX] under the assumption that the genus- $g$  generating function is a function of derivatives of genus-0 generating function:

$$\langle\langle \tau_{n+3g-1}(\gamma_\alpha) \rangle\rangle_g = \sum_{j=0}^{3g-2} \langle\langle \tau_{n+3g-2-j}(\gamma_\alpha) \gamma_\beta \rangle\rangle_0 \Gamma_j^\beta$$

where  $\Gamma_0^\beta = \langle\langle \gamma^\beta \rangle\rangle_g$  and

$$\Gamma_j^\beta = \langle\langle \tau_j(\gamma^\beta) \rangle\rangle_g - \sum_{k=0}^{j-1} \langle\langle \tau_k(\gamma^\beta) \gamma_\mu \rangle\rangle_0 \Gamma_{j-1-k}^\mu.$$

An easy induction on  $j$  shows

$$\langle\langle T^j(\gamma^\beta) \rangle\rangle_g = \Gamma_j^\beta$$

and the similar induction also shows that this topological recursion relation is precisely

$$\langle\langle T^{3g-1}(\mathcal{W}) \rangle\rangle_g = 0 \quad (25)$$

for all vector field  $\mathcal{W}$ . For  $g = 1$  and  $2$ , this equation follows from equations (11), (14), and (19). For general  $g$ , it follows from  $\psi_1^{3g-1} = 0$  on  $\overline{\mathcal{M}}_{g,1}$  since the complex dimension of  $\overline{\mathcal{M}}_{g,1}$  is  $3g - 2$ . This was first observed by Getzler.

To apply these topological recursion relations, we observe that

$$\mathcal{W} = T^k(\tau_-^k(\mathcal{W})) + \sum_{i=0}^{k-1} T^i(\overline{\tau_-^i(\mathcal{W})}) \quad (26)$$

for any vector field  $\mathcal{W}$  and  $k \geq 1$ . This formula is obtained by repeatedly applying equation (6). Since  $\overline{\tau_-^i(\mathcal{W})}$  is a primary vector field, the descendant level of  $T^i(\overline{\tau_-^i(\mathcal{W})})$  is at most  $i$ . In applications, we can choose  $k$  large enough to apply the suitable topological recursion relations. For example, to apply equation (24), we would choose  $k = g$ . To apply equation (25), we would choose  $k = 3g - 1$ .

### 3 Quantum powers of the Euler vector field

We now turn to the application of the quantum product on the big phase space to the Virasoro conjecture. As pointed in [LT], the most important vector field in studying the Virasoro conjecture is the *Euler vector field*, which is defined by

$$\mathcal{X} := - \sum_{m,\alpha} (m + b_\alpha - b_1 - 1) \tilde{t}_m^\alpha \tau_m(\gamma_\alpha) - \sum_{m,\alpha,\beta} \mathcal{C}_\alpha^\beta \tilde{t}_m^\alpha \tau_{m-1}(\gamma_\beta), \quad (27)$$

where

$$b_\alpha = (\text{holomorphic dimension of } \gamma_\alpha) - \frac{1}{2}(\text{complex dimension of } V) + \frac{1}{2}$$

and the matrix  $\mathcal{C} = (\mathcal{C}_\alpha^\beta)$  is defined by  $c_1(V) \cup \gamma_\alpha = \mathcal{C}_\alpha^\beta \gamma_\beta$ . For compact symplectic manifolds, the holomorphic dimension of  $\gamma_\alpha$  can be replaced by a half of the real dimension of  $\gamma_\alpha$  in the definition of  $b_\alpha$ . Moreover, the basis  $\{\gamma_1, \dots, \gamma_N\}$  of  $H^*(V, \mathbb{C})$  can be chosen in a way such that the following holds: If  $\eta^{\alpha\beta} \neq 0$  or  $\eta_{\alpha\beta} \neq 0$ , then  $b_\alpha = 1 - b_\beta$ ,  $\mathcal{C}_\alpha^\beta \neq 0$  implies  $b_\beta = 1 + b_\alpha$ , and  $\mathcal{C}_{\alpha\beta} \neq 0$  implies  $b_\beta = -b_\alpha$ .

The Euler vector field satisfies the following *quasi-homogeneity equation*

$$\langle\langle \mathcal{X} \rangle\rangle_g = 2(b_1 + 1)(1 - g)F_g + \frac{1}{2}\delta_{g,0} \sum_{\alpha,\beta} \mathcal{C}_{\alpha\beta} t_0^\alpha t_0^\beta - \frac{1}{24}\delta_{g,1} \int_V c_1(V) \cup c_{d-1}(V). \quad (28)$$

The genus-0 quasi-homogeneity equation implies (cf. [LT, Lemma 1.4 (3)])

$$\begin{aligned} & \left\langle \left\langle \tau_m(\gamma_\alpha) \mathcal{X} \tau_n(\gamma^\beta) \right\rangle \right\rangle_0 \\ &= \delta_{m,0} \delta_{n,0} \mathcal{C}_\alpha^\beta + (m + n + b_\alpha + 1 - b_\beta) \left\langle \left\langle \tau_m(\gamma_\alpha) \tau_n(\gamma^\beta) \right\rangle \right\rangle_0 \\ & \quad + \sum_\mu \mathcal{C}_\alpha^\mu \left\langle \left\langle \tau_{m-1}(\gamma_\mu) \tau_n(\gamma^\beta) \right\rangle \right\rangle_0 + \sum_\mu \mathcal{C}_\mu^\beta \left\langle \left\langle \tau_m(\gamma_\alpha) \tau_{n-1}(\gamma^\mu) \right\rangle \right\rangle_0. \end{aligned} \quad (29)$$

A special case of this formula is the following

$$\left\langle \left\langle \gamma_\alpha \mathcal{X} \gamma^\beta \right\rangle \right\rangle_0 = \mathcal{C}_\alpha^\beta + (b_\alpha + 1 - b_\beta) \left\langle \left\langle \gamma_\alpha \gamma^\beta \right\rangle \right\rangle_0.$$

Therefore for any vector field  $\mathcal{W}$ ,

$$\mathcal{W} \left\langle \left\langle \gamma_\alpha \mathcal{X} \gamma^\beta \right\rangle \right\rangle_0 = (b_\alpha + 1 - b_\beta) \left\langle \left\langle \gamma_\alpha \mathcal{W} \gamma^\beta \right\rangle \right\rangle_0. \quad (30)$$

An immediate application of this formula is the following Virasoro type relation among the quantum powers of the Euler vector fields:

**Theorem 3.1** *For  $m, k \geq 0$ , if  $\mathcal{W} \sim \mathcal{X}^k$  and  $\mathcal{V} \sim \mathcal{X}^m$ , then*

$$[\mathcal{W}, \mathcal{V}] \sim (m - k) \mathcal{X}^{m+k-1}.$$

Here (and thereafter)  $\mathcal{X}^0$  is understood as  $\mathcal{S}$ .

**Proof:** If  $\mathcal{W} \sim \mathcal{X}^k$ , then

$$\langle\langle \gamma_\alpha \mathcal{W} \gamma^\beta \rangle\rangle_0 = \langle\langle \gamma_\alpha \mathcal{X}^k \gamma^\beta \rangle\rangle_0$$

for all  $\alpha, \beta$ . Let  $M$  be the  $N \times N$  matrix whose  $(\alpha, \beta)$  entry is  $\langle\langle \gamma_\alpha \mathcal{X} \gamma^\beta \rangle\rangle_0$  and  $D$  be the diagonal matrix whose diagonal entries are  $b_1, \dots, b_N$ . Then

$$\langle\langle \gamma_\alpha \mathcal{X}^k \gamma^\beta \rangle\rangle_0 = \langle\langle \gamma_\alpha \mathcal{X} \gamma^{\mu_1} \rangle\rangle_0 \langle\langle \gamma_{\mu_1} \mathcal{X} \gamma^{\mu_2} \rangle\rangle_0 \dots \langle\langle \gamma_{\mu_{k-1}} \mathcal{X} \gamma^\beta \rangle\rangle_0 = (M^k)_\alpha^\beta. \quad (31)$$

Therefore, by equation (30), we have

$$\mathcal{W} M = D M^k - M^k D + M^k,$$

and consequently for any integer  $m \geq 0$ ,

$$\begin{aligned} \mathcal{W} M^m &= \sum_{j=1}^m M^{j-1} (D M^k - M^k D + M^k) M^{m-j} \\ &= m M^{m+k-1} + \sum_{j=1}^m M^{j-1} D M^{m+k-j} - \sum_{j=1}^m M^{k+j-1} D M^{m-j}. \end{aligned}$$

Hence, if  $\mathcal{W} \sim \mathcal{X}^k$  and  $\mathcal{V} \sim \mathcal{X}^m$ , then

$$\mathcal{W} M^m - \mathcal{V} M^k = (m - k) M^{m+k-1}. \quad (32)$$

Since every vector on the big phase space is equivalent to a primary vector field, to prove the theorem, it suffices to show

$$\gamma_\alpha \bullet [\mathcal{W}, \mathcal{V}] = (m - k) \gamma_\alpha \bullet \mathcal{X}^{m+k-1}$$

for all  $\alpha$ . Since  $\gamma_\alpha$  is a constant vector field, by equation (8), we have

$$\begin{aligned} \gamma_\alpha \bullet [\mathcal{W}, \mathcal{V}] &= \gamma_\alpha \bullet \nabla_{\mathcal{W}} \mathcal{V} - \gamma_\alpha \bullet \nabla_{\mathcal{V}} \mathcal{W} \\ &= \nabla_{\mathcal{W}} (\gamma_\alpha \bullet \mathcal{V}) - \nabla_{\mathcal{V}} (\gamma_\alpha \bullet \mathcal{W}) \\ &= \nabla_{\mathcal{W}} (\gamma_\alpha \bullet \mathcal{X}^m) - \nabla_{\mathcal{V}} (\gamma_\alpha \bullet \mathcal{X}^k). \end{aligned}$$

Since for any  $m \geq 0$ ,

$$\gamma_\alpha \bullet \mathcal{X}^m = (M^m)_\alpha^\beta \gamma_\beta$$

where  $(M^m)_\alpha^\beta$  is the  $(\alpha, \beta)$  entry of  $M^m$ , the theorem follows from equation (32).  $\square$

Since  $\overline{\mathcal{X}}^k \sim \mathcal{X}^k$  for all  $k$  and Lie brackets of primary fields are always primary fields, we have

**Corollary 3.2**

$$[\overline{\mathcal{X}}^k, \overline{\mathcal{X}}^m] = (m - k) \overline{\mathcal{X}}^{m+k-1}$$

for all  $m, k \geq 0$ .

We also have

**Corollary 3.3** *For  $m, k \geq 0$  and  $\{m, k\} \neq \{0, 2\}$ ,*

$$[\mathcal{X}^k, \mathcal{X}^m] = (m - k) \mathcal{X}^{m+k-1}.$$

*Note that  $[\mathcal{S}, \mathcal{X}^2] \neq 2\mathcal{X}$  since  $[\mathcal{S}, \mathcal{X}^2]$  is a primary vector field but  $\mathcal{X}$  contains descendant vector fields. In this case we have*

$$[\mathcal{S}, \mathcal{X}^2] = 2 \overline{\mathcal{X}}.$$

**Proof:** A straightforward computation shows that

$$[\mathcal{S}, \mathcal{X}] = \mathcal{S}.$$

If  $\{m, k\} \neq \{0, 1\}$ , then  $[\mathcal{X}^k, \mathcal{X}^m]$  is a primary vector field. So

$$[\mathcal{X}^k, \mathcal{X}^m] = \mathcal{S} \bullet [\mathcal{X}^k, \mathcal{X}^m] = (m - k) \overline{\mathcal{X}}^{m+k-1}$$

by Theorem 3.1. In particular,

$$[\mathcal{S}, \mathcal{X}^2] = 2 \overline{\mathcal{X}}.$$

If in addition  $\{m, k\} \neq \{0, 2\}$ , then  $(m - k) \mathcal{X}^{m+k-1}$  is also a primary vector field. So

$$(m - k) \overline{\mathcal{X}}^{m+k-1} = (m - k) \mathcal{X}^{m+k-1}.$$

This proves the corollary.  $\square$

Another application of the quasi-homogeneity equation is the following

**Lemma 3.4** *For any vector field  $\mathcal{W}$  and integer  $k \geq 0$ ,*

$$\nabla_{T(\mathcal{W})} \overline{\mathcal{X}}^k = -\mathcal{W} \bullet \overline{\mathcal{X}}^k.$$

**Proof:** By formula (30) and (31),

$$T(\mathcal{W}) \left\langle \left\langle \gamma_\alpha \mathcal{X}^k \gamma^\beta \right\rangle \right\rangle_0 = 0$$

since  $T(\mathcal{W}) \sim 0$ . This formula is also true for  $k = 0$  due to the string equation. Therefore, by Lemma 1.8,

$$\begin{aligned} \nabla_{T(\mathcal{W})} \overline{\mathcal{X}}^k &= \nabla_{T(\mathcal{W})} \left\{ \left\langle \mathcal{S} \mathcal{S} \gamma^\alpha \right\rangle_0 \left\langle \left\langle \gamma_\alpha \mathcal{X}^k \gamma^\beta \right\rangle \right\rangle_0 \gamma_\beta \right\} \\ &= \{T(\mathcal{W}) \left\langle \mathcal{S} \mathcal{S} \gamma^\alpha \right\rangle_0\} \left\langle \left\langle \gamma_\alpha \mathcal{X}^k \gamma^\beta \right\rangle \right\rangle_0 \gamma_\beta \\ &= -\left\langle \mathcal{W} \mathcal{S} \gamma^\alpha \right\rangle_0 \left\langle \left\langle \gamma_\alpha \mathcal{X}^k \gamma^\beta \right\rangle \right\rangle_0 \gamma_\beta \\ &= -\mathcal{W} \bullet \overline{\mathcal{X}}^k. \end{aligned}$$

$\square$

**Corollary 3.5**

$$[T(\overline{\mathcal{X}}^m), T(\overline{\mathcal{X}}^k)] = 0$$

for all  $m, k \geq 0$ .

**Proof:** By Lemma 1.5,

$$\begin{aligned} [T(\overline{\mathcal{X}}^m), T(\overline{\mathcal{X}}^k)] &= \nabla_{T(\overline{\mathcal{X}}^m)} T(\overline{\mathcal{X}}^k) - \nabla_{T(\overline{\mathcal{X}}^k)} T(\overline{\mathcal{X}}^m) \\ &= T(\nabla_{T(\overline{\mathcal{X}}^m)} \overline{\mathcal{X}}^k) - T(\nabla_{T(\overline{\mathcal{X}}^k)} \overline{\mathcal{X}}^m) \\ &= T(-\overline{\mathcal{X}}^{m+k}) - T(-\overline{\mathcal{X}}^{k+m}) = 0. \end{aligned}$$

□

**Corollary 3.6** For any vector field  $\mathcal{W}$  and integer  $k \geq 0$ ,

$$\nabla_{T(\mathcal{W})}(\tau_-(\mathcal{S}) \bullet \overline{\mathcal{X}}^k) = -\tau_-(\mathcal{W}) \bullet \overline{\mathcal{X}}^k.$$

**Proof:** By Lemma 1.5 and 1.8,

$$\nabla_{T(\mathcal{W})}(\tau_-(\mathcal{S})) = \tau_-(\nabla_{T(\mathcal{W})}\mathcal{S}) = -\tau_-(\mathcal{W}).$$

Therefore by equation (8), Corollary 1.6 and Theorem 3.4,

$$\begin{aligned} \nabla_{T(\mathcal{W})}(\tau_-(\mathcal{S}) \bullet \overline{\mathcal{X}}^k) &= (\nabla_{T(\mathcal{W})}(\tau_-(\mathcal{S}))) \bullet \overline{\mathcal{X}}^k + \tau_-(\mathcal{S}) \bullet \nabla_{T(\mathcal{W})} \overline{\mathcal{X}}^k \\ &\quad + \left\langle\left\langle T(\mathcal{W}) \tau_-(\mathcal{S}) \overline{\mathcal{X}}^k \gamma^\alpha \right\rangle\right\rangle_0 \gamma_\alpha \\ &= (-\tau_-(\mathcal{W})) \bullet \overline{\mathcal{X}}^k + \tau_-(\mathcal{S}) \bullet (-\mathcal{W} \bullet \overline{\mathcal{X}}^k) \\ &\quad + \mathcal{W} \bullet \tau_-(\mathcal{S}) \bullet \overline{\mathcal{X}}^k \\ &= -\tau_-(\mathcal{W}) \bullet \overline{\mathcal{X}}^k \end{aligned}$$

□

So far we have only used very special cases of equation (29). To take full advantage of this equation, we need to introduce two operations on the space of vector fields.

**Definition 3.7** For  $\mathcal{W} = \sum_{m,\alpha} f_{m,\alpha} \tau_m(\gamma_\alpha)$  and  $\mathcal{V} = \sum_{m,\alpha} g_{m,\alpha} \tau_m(\gamma_\alpha)$ , define

$$\mathcal{W} * \mathcal{V} = \sum_{m,\alpha} f_{m,\alpha} g_{m,\alpha} \tau_m(\gamma_\alpha)$$

and

$$C(\mathcal{W}) = \sum_{m,\alpha,\beta} f_{m,\alpha} \mathcal{C}_\alpha^\beta \tau_m(\gamma_\beta).$$

We can think of “ $*$ ” as a commutative and associative product on the space of vector fields with the identity

$$\mathcal{Z} := \sum_{m,\alpha} \tau_m(\gamma_\alpha). \quad (33)$$

Another important vector field for this product is

$$\mathcal{G} := \sum_{m,\alpha} (m + b_\alpha) \tau_m(\gamma_\alpha). \quad (34)$$

Using this vector field, we can define another linear transformation on the space of vector fields:

**Definition 3.8** *For any vector field  $\mathcal{W}$ , define*

$$R(\mathcal{W}) = \mathcal{G} * T(\mathcal{W}) + C(\mathcal{W}).$$

We then have the following

**Theorem 3.9** *For any vector field  $\mathcal{W}$ ,*

$$\mathcal{W} \bullet \mathcal{X} = \mathcal{S} \bullet R(\mathcal{W}) = \overline{R(\mathcal{W})}.$$

**Proof:** Let  $\mathcal{W} = \sum_{m,\alpha} f_{m,\alpha} \tau_m(\gamma_\alpha)$ . Then

$$\mathcal{W} \bullet \mathcal{X} = \sum_{m,\alpha,\mu} f_{m,\alpha} \langle\langle \tau_m(\gamma_\alpha) \mathcal{X} \gamma^\mu \rangle\rangle_0 \gamma_\mu.$$

Applying equation (29) to each 3-point functions on the right hand side, we have

$$\begin{aligned} \mathcal{W} \bullet \mathcal{X} &= \langle\langle \{\mathcal{G} * \mathcal{W}\} \gamma^\mu \rangle\rangle_0 \gamma_\mu + (1 - b_\mu) \langle\langle \mathcal{W} \gamma^\mu \rangle\rangle_0 \gamma_\mu \\ &\quad + \langle\langle \{\tau_-(C(\mathcal{W}))\} \gamma^\mu \rangle\rangle_0 \gamma_\mu + \sum_\alpha f_{0,\alpha} \mathcal{C}_\alpha^\mu \gamma_\mu. \end{aligned}$$

By equation (4), we have

$$\mathcal{W} \bullet \mathcal{X} = \mathcal{S} \bullet \{\tau_+(\mathcal{G} * \mathcal{W})\} + (1 - b_\mu) \langle\langle \mathcal{S} \tau_+(\mathcal{W}) \gamma^\mu \rangle\rangle_0 \gamma_\mu + \mathcal{S} \bullet C(\mathcal{W}).$$

On the other hand, it is straightforward to check that

$$\tau_+(\mathcal{G} * \mathcal{W}) = \mathcal{G} * \tau_+(\mathcal{W}) - \tau_+(\mathcal{W})$$

for any vector field  $\mathcal{W}$ . Therefore

$$\begin{aligned} \mathcal{W} \bullet \mathcal{X} &= \mathcal{S} \bullet \{\mathcal{G} * \tau_+(\mathcal{W})\} - b_\mu \langle\langle \mathcal{S} \tau_+(\mathcal{W}) \gamma^\mu \rangle\rangle_0 \gamma_\mu + \mathcal{S} \bullet C(\mathcal{W}) \\ &= \mathcal{S} \bullet \{\mathcal{G} * T(\mathcal{W}) + \mathcal{G} * (\mathcal{S} \bullet \tau_+(\mathcal{W}))\} - \mathcal{G} * (\mathcal{S} \bullet \tau_+(\mathcal{W})) + \mathcal{S} \bullet C(\mathcal{W}). \end{aligned}$$

The middle terms in the last expression are cancelled since  $\mathcal{G} * (\mathcal{S} \bullet \tau_+(\mathcal{W}))$  is a primary vector field. This proves the theorem.  $\square$

**Corollary 3.10** *If  $\mathcal{W} \sim \mathcal{V}$ , then  $R(\mathcal{W}) \sim R(\mathcal{V})$ .*

**Proof:**  $R(\mathcal{W}) \sim R(\mathcal{V})$  if and only if  $\mathcal{S} \bullet R(\mathcal{W}) = \mathcal{S} \bullet R(\mathcal{V})$ , which indeed follows from Theorem 3.9.  $\square$

The following formula will be useful later.

**Lemma 3.11** *For any vector field  $\mathcal{W}$ ,*

$$R(\tau_-(\mathcal{W})) = \tau_-(R(\mathcal{W})) - \mathcal{G} * \overline{\mathcal{W}} - \mathcal{W}.$$

**Proof:** It is straightforward to check that

$$\tau_-(\mathcal{G} * \mathcal{W}) = \mathcal{G} * \tau_-(\mathcal{W}) + \tau_-(\mathcal{W})$$

and

$$\tau_-(C(\mathcal{W})) = C(\tau_-(\mathcal{W}))$$

for any vector field  $\mathcal{W}$ . Therefore

$$\begin{aligned} \tau_-(R(\mathcal{W})) &= \mathcal{G} * \mathcal{W} + \mathcal{W} + C(\tau_-(\mathcal{W})) \\ &= \mathcal{G} * (T(\tau_-(\mathcal{W})) + \overline{\mathcal{W}}) + \mathcal{W} + C(\tau_-(\mathcal{W})) \\ &= R(\tau_-(\mathcal{W})) + \mathcal{G} * \overline{\mathcal{W}} + \mathcal{W}. \end{aligned}$$

The lemma follows.  $\square$

Since  $T(\mathcal{S}) = \mathcal{D}$ , it is straightforward to check that

$$R(\mathcal{S}) = \mathcal{X} + (b_1 + 1)\mathcal{D} \tag{35}$$

So by Theorem 3.9 and Lemma 3.11,

$$\begin{aligned} \mathcal{X} \bullet \tau_-(\mathcal{S}) &= \mathcal{S} \bullet R(\tau_-(\mathcal{S})) \\ &= \mathcal{S} \bullet (\tau_-(R(\mathcal{S})) - \mathcal{G} * \mathcal{S}^2 - \mathcal{S}) \\ &= \mathcal{S} \bullet (\tau_-(\mathcal{X} + (b_1 + 1)\mathcal{D}) - \mathcal{G} * \mathcal{S}^2 - \mathcal{S}) \\ &= \mathcal{S} \bullet \tau_-(\mathcal{X}) + b_1 \mathcal{S}^2 - \mathcal{G} * \mathcal{S}^2. \end{aligned}$$

Equivalently, we have

$$\mathcal{S} \bullet \tau_-(\mathcal{X}) = \mathcal{X} \bullet \tau_-(\mathcal{S}) - b_1 \mathcal{S}^2 + \mathcal{G} * \mathcal{S}^2. \tag{36}$$

Another useful formula for the operator  $R$  is the following

**Lemma 3.12** *For any vector fields  $\mathcal{W}$  and  $\mathcal{V}$ ,*

$$\nabla_{\mathcal{V}}(R(\mathcal{W})) = R(\nabla_{\mathcal{V}}\mathcal{W}) - \mathcal{G} * (\mathcal{V} \bullet \mathcal{W}).$$

**Proof:** It is straightforward to check that

$$\nabla_{\mathcal{V}}(\mathcal{G} * \mathcal{W}) = \mathcal{G} * (\nabla_{\mathcal{V}}\mathcal{W}) \quad \text{and} \quad \nabla_{\mathcal{V}}(C(\mathcal{W})) = C(\nabla_{\mathcal{V}}\mathcal{W}) \quad (37)$$

for any vector fields  $\mathcal{W}$  and  $\mathcal{V}$ . Therefore

$$\begin{aligned} \nabla_{\mathcal{V}}(R(\mathcal{W})) &= \nabla_{\mathcal{V}}(\mathcal{G} * T(\mathcal{W}) + C(\mathcal{W})) \\ &= \mathcal{G} * \nabla_{\mathcal{V}}T(\mathcal{W}) + C(\nabla_{\mathcal{V}}\mathcal{W}) \\ &= \mathcal{G} * (T(\nabla_{\mathcal{V}}\mathcal{W}) - \mathcal{V} \bullet \mathcal{W}) + C(\nabla_{\mathcal{V}}\mathcal{W}) \\ &= R(\nabla_{\mathcal{V}}\mathcal{W}) - \mathcal{G} * (\mathcal{V} \bullet \mathcal{W}). \end{aligned}$$

□

The quasi-homogeneity equation for all genera implies the following rule for eliminating the Euler vector field from a correlation function:

**Lemma 3.13**

$$\begin{aligned} \langle\langle \mathcal{X} \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g &= \sum_{i=1}^k \langle\langle \mathcal{W}_1 \cdots \{\tau_- R(\mathcal{W}_i)\} \cdots \mathcal{W}_k \rangle\rangle_g \\ &\quad - \{(2g + k - 2)b_1 + 2(g + k - 1)\} \langle\langle \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g \\ &\quad + \delta_{g,0} \nabla_{\mathcal{W}_1, \dots, \mathcal{W}_k}^k \left( \frac{1}{2} \mathcal{C}_{\alpha\beta} t_0^\alpha t_0^\beta \right) \end{aligned}$$

for any vector fields  $\mathcal{W}_1, \dots, \mathcal{W}_k$  and  $k \geq 1$ . Here  $\nabla_{\mathcal{W}_1, \dots, \mathcal{W}_k}^k$  is the  $k$ -th covariant derivative. Note that if  $g > 0$  or  $k > 2$ , the last term on the right hand side vanishes.

**Proof:** It is straightforward to check that

$$\begin{aligned} \nabla_{\mathcal{W}}\mathcal{X} &= -\mathcal{G} * \mathcal{W} + (b_1 + 1)\mathcal{W} - C(\tau_-(\mathcal{W})) \\ &= -R(\tau_-(\mathcal{W})) - \mathcal{G} * \overline{\mathcal{W}} + (b_1 + 1)\mathcal{W} \end{aligned}$$

for any vector field  $\mathcal{W}$ . By Lemma 3.11, we also have

$$\nabla_{\mathcal{W}}\mathcal{X} = -\tau_- R(\mathcal{W}) + (b_1 + 2)\mathcal{W}. \quad (38)$$

It suffices to prove the lemma for parallel vector fields  $\mathcal{W}_i$  since both sides of the equation are tensors in these vector fields. We observed that if  $\mathcal{W}$  is parallel, i.e.  $\nabla_{\mathcal{V}}\mathcal{W} = 0$  for all  $\mathcal{V}$ , then  $\tau_- R(\mathcal{W})$  is also parallel. In fact, by Lemma 1.5 and Lemma 3.12

$$\nabla_{\mathcal{V}}\tau_- R(\mathcal{W}) = \tau_- R(\nabla_{\mathcal{V}}\mathcal{W}) - \tau_-(\mathcal{G} * (\mathcal{V} \bullet \mathcal{W})) = 0$$

since  $\mathcal{W}$  is parallel and  $\mathcal{G} * (\mathcal{V} \bullet \mathcal{W})$  is a primary vector field. The lemma then follows from repeatedly taking derivatives of equation (28) and using equation (7). □

In case that  $\mathcal{W}$  is a primary vector field,  $\tau_-(\mathcal{W}) = 0$ . Therefore by Lemma 3.11,

$$\tau_- R(\mathcal{W}) = \mathcal{G} * \mathcal{W} + \mathcal{W}.$$



Hence the formula in Lemma 3.13 can be written as

$$\begin{aligned}
\langle\langle \mathcal{X} \mathcal{W}_1 \dots \mathcal{W}_k \rangle\rangle_g &= \sum_{i=1}^k \langle\langle \mathcal{W}_1 \dots \{\mathcal{G} * \mathcal{W}_i\} \dots \mathcal{W}_k \rangle\rangle_g \\
&\quad - \{k-2+2g+(2g+k-2)b_1\} \langle\langle \mathcal{W}_1 \dots \mathcal{W}_k \rangle\rangle_g \\
&\quad + \delta_{g,0} \nabla_{\mathcal{W}_1, \dots, \mathcal{W}_k}^k \left( \frac{1}{2} \mathcal{C}_{\alpha\beta} t_0^\alpha t_0^\beta \right)
\end{aligned} \tag{39}$$

for all primary fields  $\mathcal{W}_1, \dots, \mathcal{W}_k$  and  $k \geq 2$ .

An immediate consequence of Lemma 3.13 is the following

**Lemma 3.14**

$$\begin{aligned}
\langle\langle \mathcal{X} \mathcal{X} \mathcal{X} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha &= 2\mathcal{X}^3 \bullet \tau_-(\mathcal{S}) - 3b_1 \mathcal{X}^2 \\
&\quad + 2\mathcal{X}^2 \bullet (\mathcal{G} * \overline{\mathcal{S}}) + 2\mathcal{X} \bullet (\mathcal{G} * \overline{\mathcal{X}}) - \mathcal{G} * \mathcal{X}^2, \\
\langle\langle \mathcal{X} \mathcal{X} \overline{\mathcal{X}}^k \gamma^\alpha \rangle\rangle_0 \gamma_\alpha &= \mathcal{X}^{k+2} \bullet \tau_-(\mathcal{S}) - 2b_1 \overline{\mathcal{X}}^{k+1} + \mathcal{X}^{k+1} \bullet (\mathcal{G} * \overline{\mathcal{S}}) \\
&\quad + \mathcal{X}^k \bullet (\mathcal{G} * \overline{\mathcal{X}}) + \mathcal{X} \bullet (\mathcal{G} * \overline{\mathcal{X}}^k) - \mathcal{G} * \mathcal{X}^2, \\
\langle\langle \mathcal{X} \overline{\mathcal{X}}^m \overline{\mathcal{X}}^k \gamma^\alpha \rangle\rangle_0 \gamma_\alpha &= -b_1 \overline{\mathcal{X}}^{m+k} - \mathcal{G} * \overline{\mathcal{X}}^{m+k} \\
&\quad + \mathcal{X}^k \bullet (\mathcal{G} * \overline{\mathcal{X}}^m) + \mathcal{X}^m \bullet (\mathcal{G} * \overline{\mathcal{X}}^k),
\end{aligned}$$

where  $m, k \geq 0$ .

**Proof:** The last formula is a direct consequence of equation (39). Since by equation (6),

$$\mathcal{X} = T(\tau_-(\mathcal{X})) + \overline{\mathcal{X}},$$

corollary 1.6 implies that

$$\begin{aligned}
&\langle\langle \mathcal{X} \mathcal{X} \overline{\mathcal{X}}^k \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\
&= \langle\langle \mathcal{X} T(\tau_-(\mathcal{X})) (\overline{\mathcal{X}}^k) \gamma^\alpha \rangle\rangle_0 \gamma_\alpha + \langle\langle \mathcal{X} \overline{\mathcal{X}} \overline{\mathcal{X}}^k \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\
&= \mathcal{X}^{k+1} \bullet \tau_-(\mathcal{X}) + \langle\langle \mathcal{X} \overline{\mathcal{X}} \overline{\mathcal{X}}^k \gamma^\alpha \rangle\rangle_0 \gamma_\alpha
\end{aligned}$$

and

$$\begin{aligned}
&\langle\langle \mathcal{X} \mathcal{X} \mathcal{X} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\
&= \langle\langle \mathcal{X} T(\tau_-(\mathcal{X})) \mathcal{X} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha + \langle\langle \mathcal{X} \overline{\mathcal{X}} \mathcal{X} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\
&= \mathcal{X}^2 \bullet \tau_-(\mathcal{X}) + \langle\langle \mathcal{X} \overline{\mathcal{X}} \mathcal{X} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\
&= 2\mathcal{X}^2 \bullet \tau_-(\mathcal{X}) + \langle\langle \mathcal{X} \overline{\mathcal{X}} \overline{\mathcal{X}} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha
\end{aligned}$$

Therefore the first two equations in the lemma follows from the third equation and equation (36).  $\square$

If a 4-point function does not contain  $\mathcal{X}$ , but contains  $\mathcal{X}^k$  with  $k \geq 2$ , it can be computed by using the following lemma to create 4-point functions involving  $\mathcal{X}$ .

**Lemma 3.15** *For any vector field  $\mathcal{W}$ ,  $\mathcal{V}_1$ ,  $\mathcal{V}_2$ ,  $\mathcal{V}_3$ , and  $k \geq 2$ ,*

$$\begin{aligned} \langle\langle \mathcal{W}^k \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \rangle\rangle_0 &= - \sum_{i=1}^{k-1} \langle\langle \mathcal{W} \mathcal{W}^{k-i} \{ \mathcal{V}_1 \bullet \mathcal{V}_2 \bullet \mathcal{W}^{i-1} \} \mathcal{V}_3 \rangle\rangle_0 \\ &\quad + \sum_{i=2}^{k-1} \langle\langle \mathcal{W} \{ \mathcal{W}^{k-i} \bullet \mathcal{V}_1 \} \{ \mathcal{V}_2 \bullet \mathcal{W}^{i-1} \} \mathcal{V}_3 \rangle\rangle_0 \\ &\quad + \langle\langle \mathcal{W} \{ \mathcal{W}^{k-1} \bullet \mathcal{V}_1 \} \mathcal{V}_2 \mathcal{V}_3 \rangle\rangle_0 + \langle\langle \mathcal{W} \mathcal{V}_1 \{ \mathcal{V}_2 \bullet \mathcal{W}^{k-1} \} \mathcal{V}_3 \rangle\rangle_0. \end{aligned}$$

This lemma follows from the following form of the first derivatives of the generalized WDVV equation

$$\begin{aligned} \langle\langle \{ \mathcal{W}_1 \bullet \mathcal{W}_2 \} \mathcal{W}_3 \mathcal{W}_4 \mathcal{W}_5 \rangle\rangle_0 &= \langle\langle \{ \mathcal{W}_1 \bullet \mathcal{W}_3 \} \mathcal{W}_2 \mathcal{W}_4 \mathcal{W}_5 \rangle\rangle_0 \\ &\quad + \langle\langle \mathcal{W}_1 \mathcal{W}_3 \{ \mathcal{W}_2 \bullet \mathcal{W}_4 \} \mathcal{W}_5 \rangle\rangle_0 \\ &\quad - \langle\langle \mathcal{W}_1 \mathcal{W}_2 \{ \mathcal{W}_3 \bullet \mathcal{W}_4 \} \mathcal{W}_5 \rangle\rangle_0 \end{aligned} \quad (40)$$

for any vector fields  $\mathcal{W}_1, \dots, \mathcal{W}_5$ . A similar lemma on the small phase space was proved in [L1]. The same proof also works here on the big phase space.

**Corollary 3.16** *For all primary vector fields  $\mathcal{W}$ ,  $\mathcal{V}$ , and  $k \geq 0$ ,*

$$\begin{aligned} \langle\langle \overline{\mathcal{X}}^k \mathcal{W} \mathcal{V} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha &= -\overline{\mathcal{X}}^k \bullet \mathcal{W} \bullet \mathcal{V} \bullet \tau_-(\mathcal{S}) \\ &\quad + \sum_{i=1}^k \left\{ (\mathcal{G} * (\overline{\mathcal{X}}^{k-i} \bullet \mathcal{W})) \bullet \mathcal{V} \bullet \overline{\mathcal{X}}^{i-1} + \overline{\mathcal{X}}^{k-i} \bullet \mathcal{W} \bullet (\mathcal{G} * (\mathcal{V} \bullet \overline{\mathcal{X}}^{i-1})) \right. \\ &\quad \left. - (\mathcal{G} * \overline{\mathcal{X}}^{k-i}) \bullet \mathcal{W} \bullet \mathcal{V} \bullet \overline{\mathcal{X}}^{i-1} - \overline{\mathcal{X}}^{k-i} \bullet (\mathcal{G} * (\mathcal{W} \bullet \mathcal{V} \bullet \overline{\mathcal{X}}^{i-1})) \right\} \end{aligned}$$

**Proof:** Since  $\overline{\mathcal{S}} = \mathcal{S} - T(\tau_-(\mathcal{S}))$ , by Corollary 1.6 and Lemma 1.8,

$$\begin{aligned} &\langle\langle \overline{\mathcal{S}} \mathcal{W} \mathcal{V} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\ &= \langle\langle \mathcal{S} \mathcal{W} \mathcal{V} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha - \tau_-(\mathcal{S}) \bullet \mathcal{W} \bullet \mathcal{V} \\ &= -\tau_-(\mathcal{S}) \bullet \mathcal{W} \bullet \mathcal{V} \end{aligned}$$

if  $\mathcal{W}$  and  $\mathcal{V}$  are primary fields. This proves the corollary for  $k = 0$ . Similarly since  $\overline{\mathcal{X}} = \mathcal{X} - T(\tau_-(\mathcal{X}))$ , by Corollary 1.6,

$$\begin{aligned} &\langle\langle \overline{\mathcal{X}} \mathcal{W} \mathcal{V} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\ &= \langle\langle \mathcal{X} \mathcal{W} \mathcal{V} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha - \tau_-(\mathcal{X}) \bullet \mathcal{W} \bullet \mathcal{V}. \end{aligned}$$

By equation (39) and equation (36),

$$\begin{aligned} &\langle\langle \overline{\mathcal{X}} \mathcal{W} \mathcal{V} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\ &= -\overline{\mathcal{X}} \bullet \mathcal{W} \bullet \mathcal{V} \bullet \tau_-(\mathcal{S}) - \mathcal{W} \bullet \mathcal{V} \bullet (\mathcal{G} * \overline{\mathcal{S}}) - \mathcal{G} * (\mathcal{W} \bullet \mathcal{V}) \\ &\quad + \mathcal{W} \bullet (\mathcal{G} * \mathcal{V}) + \mathcal{V} \bullet (\mathcal{G} * \mathcal{W}). \end{aligned}$$

This proves the corollary for  $k = 1$ . For  $k \geq 2$ , the corollary follows from Lemma 3.15 and equation (39).  $\square$

In particular, we have

**Corollary 3.17**

$$\begin{aligned}
& \left\langle \left\langle \overline{\mathcal{X}}^n \overline{\mathcal{X}}^m \overline{\mathcal{X}}^k \gamma^\alpha \right\rangle \right\rangle_0 \gamma_\alpha \\
&= -\mathcal{X}^{n+m+k} \bullet \tau_-(\mathcal{S}) \\
&\quad - \sum_{i=0}^{k-1} \mathcal{X}^i \bullet (\mathcal{G} * \overline{\mathcal{X}}^{n+m+k-i-1}) - \sum_{i=n+m}^{n+m+k-1} \mathcal{X}^i \bullet (\mathcal{G} * \overline{\mathcal{X}}^{n+m+k-i-1}) \\
&\quad + \sum_{i=m}^{m+k-1} \mathcal{X}^i \bullet (\mathcal{G} * \overline{\mathcal{X}}^{n+m+k-i-1}) + \sum_{i=n}^{n+k-1} \mathcal{X}^i \bullet (\mathcal{G} * \overline{\mathcal{X}}^{n+m+k-i-1})
\end{aligned}$$

for  $n, m, k \geq 0$ .

## 4 Virasoro vector fields

In this section, we study a sequence of vector fields on the big phase space obtained from the string vector field by recursively applying the operator  $R$  defined in Definition 3.8. We will show that this sequence of vector fields satisfy the Virasoro bracket relation and arise naturally in the study of higher genus Virasoro conjecture. For this reason, we call these vector fields Virasoro vector fields.

Define the  $k$ -th *Virasoro vector field* by

$$\mathcal{L}_k := R^{k+1}(-\mathcal{S}) \quad (41)$$

for  $k \geq -1$ . Here  $\mathcal{L}_{-1}$  should be understood as  $-\mathcal{S}$ . By Theorem 3.9, these vector fields satisfy the following interesting property

**Lemma 4.1**

$$\mathcal{L}_n \sim -\mathcal{X}^{n+1}$$

for all  $n \geq -1$ .

Moreover Lemma 3.12 and Lemma 1.8 make it simple to compute the covariant derivatives of these vector fields. In fact, we have the following

**Lemma 4.2**

$$\nabla_{\mathcal{W}} \mathcal{L}_k = \tau_- R^{k+1}(\mathcal{W}) - (k+1) R^k(\mathcal{W})$$

for any vector field  $\mathcal{W}$ .

**Proof:** By Lemma 1.8

$$\nabla_{\mathcal{W}} \mathcal{L}_{-1} = -\nabla_{\mathcal{W}} \mathcal{S} = \tau_-(\mathcal{W}).$$

This proves the lemma for  $k = -1$ . We now prove the lemma by induction on  $k$ . Suppose that the lemma holds for  $k = n$ . Then by Lemma 3.12 and Lemma 3.11

$$\begin{aligned}\nabla_{\mathcal{W}}\mathcal{L}_{n+1} &= \nabla_{\mathcal{W}}R(\mathcal{L}_n) = R(\nabla_{\mathcal{W}}\mathcal{L}_n) - \mathcal{G} * (\mathcal{W} \bullet \mathcal{L}_n) \\ &= R\left\{\tau_- R^{n+1}(\mathcal{W}) - (n+1)R^n(\mathcal{W})\right\} - \mathcal{G} * (\mathcal{W} \bullet \mathcal{L}_n) \\ &= \tau_- R^{n+2}(\mathcal{W}) - \mathcal{G} * \overline{R^{n+1}(\mathcal{W})} - (n+2)R^{n+1}(\mathcal{W}) - \mathcal{G} * (\mathcal{W} \bullet \mathcal{L}_n).\end{aligned}$$

The lemma follows since by Theorem 3.9

$$\overline{R^{n+1}(\mathcal{W})} = \mathcal{X}^{n+1} \bullet \mathcal{W} = -\mathcal{W} \bullet \mathcal{L}_n.$$

□

Since covariant derivatives commute with the operator  $\tau_-$ , we have

**Corollary 4.3**

$$\nabla_{\mathcal{W}} \tau_-^m(\mathcal{L}_k) = \tau_-^{m+1} R^{k+1}(\mathcal{W}) - (k+1)\tau_-^m R^k(\mathcal{W})$$

for any vector field  $\mathcal{W}$ .

A special case of this corollary is the following

$$\nabla_{\tau_-^n(\mathcal{L}_j)} \tau_-^m(\mathcal{L}_k) = \tau_-^{m+1} R^{k+1}\tau_-^n(\mathcal{L}_j) - (k+1)\tau_-^m R^k \tau_-^n(\mathcal{L}_j).$$

We can use this formula to compute the brackets of vector fields  $\tau_-^m \mathcal{L}_k$ . For example, when  $m = n = 0$ , we have

$$\nabla_{\mathcal{L}_j} \mathcal{L}_k = \tau_- (\mathcal{L}_{k+j+1}) - (k+1)\mathcal{L}_{k+j}. \quad (42)$$

Since  $[\mathcal{L}_j, \mathcal{L}_k] = \nabla_{\mathcal{L}_j} \mathcal{L}_k - \nabla_{\mathcal{L}_k} \mathcal{L}_j$ , we have

**Corollary 4.4**

$$[\mathcal{L}_j, \mathcal{L}_k] = (j-k)\mathcal{L}_{k+j}$$

for  $k, j \geq -1$ .

In other words, the sequence of vector fields  $\{\mathcal{L}_k = R^{k+1}(-\mathcal{S}) \mid k \geq -1\}$  form a half branch of the Virasoro algebra. More generally, in view of Lemma 3.11, the more general class of vector fields  $\{\tau_-^m(\mathcal{L}_k) \mid m \geq 0, k \geq -1\}$  seems generate a  $W$ -type algebra.

In some cases, it is more convenient to use the following formula for covariant derivatives of the Virasoro vector fields.

**Lemma 4.5**

$$\nabla_{\mathcal{W}}\mathcal{L}_k = R^{k+1}\tau_-(\mathcal{W}) + \sum_{i=0}^k R^i(\mathcal{G} * (\mathcal{X}^{k-i} \bullet \mathcal{W}))$$

for any vector field  $\mathcal{W}$ .

**Proof:** This follows from Lemma 4.2 by interchanging positions of  $\tau_-$  and  $R^{k+1}$  using Lemma 3.11.  $\square$

Since  $T(\mathcal{W}) \bullet \mathcal{V} = 0$  for any vector field  $\mathcal{V}$ , one application of Lemma 4.5 is the following

$$\nabla_{T(\mathcal{W})}\mathcal{L}_k = R^{k+1}(\mathcal{W}) \quad (43)$$

for any vector field  $\mathcal{W}$ . In particular,

$$\nabla_{T(\mathcal{L}_m)}T(\mathcal{L}_k) = T(\nabla_{T(\mathcal{L}_m)}\mathcal{L}_k) = T(\mathcal{L}_{m+k+1}).$$

Therefore

$$[T(\mathcal{L}_m), T(\mathcal{L}_k)] = 0$$

for all  $m, k \geq -1$ . Moreover, the two sequences of vector fields  $\{\mathcal{L}_k \mid k \geq -1\}$  and  $\{T(\mathcal{L}_k) \mid k \geq -1\}$  together form a Lie algebra isomorphic to the Lie algebra spanned by  $\{t^m \partial_t, t^k \mid m, k \geq 0\}$  as operators on the space of functions on the unit circle, where  $t$  is the standard coordinate on the circle. The isomorphism between these two Lie algebras is given by the map

$$\mathcal{L}_k \longmapsto -t^{k+1} \partial_t, \quad T(\mathcal{L}_k) \longmapsto -t^{k+1}$$

for  $k \geq -1$ . To verify this statement, we only need to check the bracket relation

$$[T(\mathcal{L}_m), \mathcal{L}_k] = (m+1)T(\mathcal{L}_{m+k}).$$

This follows from the fact that  $\nabla_{T(\mathcal{L}_m)}\mathcal{L}_k = \mathcal{L}_{m+k+1}$  by equation (43) and

$$\begin{aligned} \nabla_{\mathcal{L}_k}T(\mathcal{L}_m) &= T(\nabla_{\mathcal{L}_k}\mathcal{L}_m) - \mathcal{L}_k \bullet \mathcal{L}_m \\ &= T(\tau_-(\mathcal{L}_{m+k+1}) - (m+1)T(\mathcal{L}_{m+k})) + \overline{\mathcal{L}_{m+k+1}} \\ &= \mathcal{L}_{m+k+1} - (m+1)T(\mathcal{L}_{m+k}) \end{aligned}$$

by equation (42), Lemma 4.1 and equation (6). Similar computations show that

$$[\mathcal{L}_k, T^n(\mathcal{L}_j)] = (T^{n-1}R^{k+1} - R^{k+1}T^{n-1} - (j+1)T^n R^k)(\mathcal{L}_j)$$

and

$$[T^m(\mathcal{L}_k), T^n(\mathcal{L}_j)] = (T^n R^{j+1}T^{m-1}R^{k+1} - T^m R^{k+1}T^{n-1}R^{j+1})(\mathcal{L}_{-1})$$

for all  $m, n \geq 1$  and  $j, k \geq -1$ . Therefore the set of vector fields  $\{T^m(\mathcal{L}_k) \mid m \geq 0, k \geq -1\}$  is closed under the Lie bracket (and consequently forms a Lie algebra) due to the following fact

**Lemma 4.6**

$$RT = TR + T^2.$$

**Proof:** For any vector field  $\mathcal{W}$ ,

$$RT(\mathcal{W}) = T\tau_-RT(\mathcal{W}) + \overline{RT(\mathcal{W})}$$

by equation (6). Since  $\overline{RT(\mathcal{W})} = \mathcal{X} \bullet T(\mathcal{W}) = 0$ , by Lemma 3.11,

$$RT(\mathcal{W}) = T\left(R\tau_-T(\mathcal{W}) + \mathcal{G} * \overline{T(\mathcal{W})} + T(\mathcal{W})\right) = TR(\mathcal{W}) + T^2(\mathcal{W}).$$

The lemma is thus proved.  $\square$

Let  $\partial_t^{-1}$  be the integral operator on the space of functions on the unit circle, where  $t$  is the standard coordinate on the circle. Then as operators,

$$t\partial_t^{-1} = \partial_t^{-1}t + \left(\partial_t^{-1}\right)^2.$$

Therefore the map

$$T^m(\mathcal{L}_k) \longmapsto -\left(\partial_t^{-1}\right)^m t^{k+1}\partial_t$$

for  $k \geq -1$  and  $m \geq 1$  defines an isomorphism between the Lie algebra  $\{T^m(\mathcal{L}_k) \mid m \geq 1, k \geq -1\}$  and the Lie algebra of integral operators  $\{t^k(\partial_t^{-1})^m \mid m, k \geq 0\}$ . Since  $\mathcal{L}_k = R^{k+1}(\mathcal{L}_{-1})$ , we see that under this isomorphism,  $R$  corresponds to multiplying by  $t$ , and  $T$  corresponds to composition by  $\partial_t^{-1}$  on the space of pseudo-differential operators on the unit circle. As a left inverse of  $T$ ,  $\tau_-$  might be expected to correspond to  $\partial_t$ . However, the commutator of  $\tau_-$  and  $R$  is not quite the same as the commutator of  $\partial_t$  and  $t$  due to the twisting given by  $\mathcal{G} * (\mathcal{S} \bullet)$  in Lemma 3.11.

To see the connection between vector fields  $\mathcal{L}_n$  with the Virasoro conjecture, we explicitly compute the first several Virasoro vector fields and obtain the following

#### Theorem 4.7

$$\begin{aligned} \mathcal{L}_{-1} &= -\mathcal{S}, \\ \mathcal{L}_0 &= -\mathcal{X} - (b_1 + 1)\mathcal{D}, \\ \mathcal{L}_1 &= \sum_{m,\alpha} (m + b_\alpha)(m + b_\alpha + 1)\tilde{t}_m^\alpha \tau_{m+1}(\gamma_\alpha) + \sum_{m,\alpha,\beta} (2m + 2b_\alpha + 1)\mathcal{C}_\alpha^\beta \tilde{t}_m^\alpha \tau_m(\gamma_\beta) \\ &\quad + \sum_{m,\alpha,\beta} (\mathcal{C}^2)_\alpha^\beta \tilde{t}_m^\alpha \tau_{m-1}(\gamma_\beta) - \sum_\alpha b_\alpha(b_\alpha - 1) \langle\langle \gamma^\alpha \rangle\rangle_0 \gamma_\alpha, \\ \mathcal{L}_2 &= \sum_{m,\alpha} (m + b_\alpha)(m + b_\alpha + 1)(m + b_\alpha + 2)\tilde{t}_m^\alpha \tau_{m+2}(\gamma_\alpha) \\ &\quad + \sum_{m,\alpha,\beta} \left\{ 3(m + b_\alpha)^2 + 6(m + b_\alpha) + 2 \right\} \mathcal{C}_\alpha^\beta \tilde{t}_m^\alpha \tau_{m+1}(\gamma_\beta) \\ &\quad + \sum_{m,\alpha,\beta} 3(m + b_\alpha + 1)(\mathcal{C}^2)_\alpha^\beta \tilde{t}_m^\alpha \tau_m(\gamma_\beta) + \sum_{m,\alpha,\beta} (\mathcal{C}^3)_\alpha^\beta \tilde{t}_m^\alpha \tau_{m-1}(\gamma_\beta) \\ &\quad - \sum_\alpha b_\alpha(b_\alpha^2 - 1) \{ \langle\langle \tau_1(\gamma_\alpha) \rangle\rangle_0 \gamma^\alpha + \langle\langle \gamma^\alpha \rangle\rangle_0 \tau_1(\gamma_\alpha) \} - \sum_{\alpha,\beta} (3b_\beta^2 - 1) \mathcal{C}_\beta^\alpha \langle\langle \gamma_\alpha \rangle\rangle_0 \gamma^\beta. \end{aligned}$$

**Proof:** The first equation is just the definition of  $\mathcal{L}_{-1}$ . The second equation is precisely equation (35). The genus-0 quasi-homogeneity equation and the dilaton equation implies (cf. [LT, Lemma 1.2 and 1.4])

$$\langle\langle \mathcal{L}_0 \gamma^\alpha \rangle\rangle_0 = -(1 - b_\alpha) \langle\langle \gamma^\alpha \rangle\rangle_0 - \mathcal{C}_\beta^\alpha t_0^\beta.$$

The genus-0  $L_1$ -constraint (cf. [LT, Formula (23)]) implies

$$\langle\langle \mathcal{L}_1 \gamma^\alpha \rangle\rangle_0 = -(1 - b_\alpha)(2 - b_\alpha) \langle\langle \tau_1(\gamma^\alpha) \rangle\rangle_0 - (3 - 2b_\alpha) \mathcal{C}_\beta^\alpha \langle\langle \gamma^\beta \rangle\rangle_0 - (\mathcal{C}^2)_\beta^\alpha t_0^\beta.$$

Using these formulas and the definitions of operators  $T$  and  $R$ , it is straightforward to check the last two equations in the lemma.  $\square$

From the formulas in Theorem 4.7, we see that  $\mathcal{L}_{-1}$  and  $\mathcal{L}_0$  are the first derivative parts of the first two Virasoro operators defined in [EHX]. These vector fields were considered in [LT]. The first derivative parts of the Virasoro operators  $L_1$  and  $L_2$  are also considered in [LT] and played important role in the proof of the genus-0 Virasoro conjecture. These vector fields are the same as the linear parts of formulas for  $\mathcal{L}_1$  and  $\mathcal{L}_2$  (omitting the finitely many terms containing genus-0 1-point functions). The formulas for  $\mathcal{L}_1$  and  $\mathcal{L}_2$  as given in Theorem 4.7 do coincide with the corresponding vector fields used in [DZ2] and [G2]. These vector fields arise naturally when studying Virasoro conjecture of genus bigger than 0. More explicitly, for  $g \geq 1$ , the genus- $g$   $L_n$ -constraint just computes  $\langle\langle \mathcal{L}_n \rangle\rangle_g$  in terms of data with genus less than  $g$ . This is true not only for  $-1 \leq n \leq 2$ . For  $n > 2$ , this follows from Corollary 4.4 since the first four Virasoro vector fields generate all others by taking iterating brackets.

We can now also give a new interpretation for Lemma 4.1. Using the formulas in Theorem 4.7, we see that the most important cases of this lemma, i.e.

$$\mathcal{L}_1 \sim -\mathcal{X}^2 \quad \text{and} \quad \mathcal{L}_2 \sim -\mathcal{X}^3, \tag{44}$$

are special cases of [LT, Equation (19) and (26)], which are crucial steps in the proof of the genus-0 Virasoro conjecture. Because of equation (2) these relations are equivalent to

$$\overline{\mathcal{L}}_1 = -\mathcal{X}^2 \quad \text{and} \quad \overline{\mathcal{L}}_2 = -\mathcal{X}^3, \tag{45}$$

which are special cases of [LT, Lemma 3.1 and Lemma 4.2] due to equation (4) and [LT, Lemma 1.4]. We notice that Lemma 4.1 also follows from equation (44) due to Theorem 3.1. As we see from [LT], equation (44) is equivalent to the second derivatives of the genus-0  $L_1$  and  $L_2$  constraints. Therefore, we can interpret Lemma 4.1 as the second derivatives of the genus-0 Virasoro conjecture. As explained in [LT], the second derivatives of the genus-0 Virasoro constraints imply the genus-0 Virasoro conjecture because of the dilaton equation. We notice that in the proof of Theorem 4.7, the formula for  $L_k$  is derived only using genus-0  $L_{k-1}$ -constraint. Therefore one can modify the proof of Theorem 4.7 to give a new proof to the genus-0 Virasoro conjecture using Lemma 4.1.

To prove higher genera Virasoro conjecture, one needs to compute  $\langle\langle \mathcal{L}_n \rangle\rangle_g$ . Applying equation (26) to  $\mathcal{L}_n$  and consider the genus- $g$  topological recursion relation, we see that

it is important to understand vector fields  $\overline{\tau_-^m(\mathcal{L}_n)}$ . For  $m = 0$ ,  $\overline{\tau_-^m(\mathcal{L}_n)} = \overline{\mathcal{L}_n} = -\overline{\mathcal{X}}^{n+1}$  by Lemma 4.1. We will see that when  $m \geq 1$ , all such vector fields can be expressed in terms of certain twisted quantum powers of the Euler vector field. We have the following

**Theorem 4.8**

$$\overline{\tau_-^m(\mathcal{L}_{n+1})} = \mathcal{X} \bullet \overline{\tau_-^m(\mathcal{L}_n)} + m \overline{\tau_-^{m-1}(\mathcal{L}_n)} + \mathcal{G} * \overline{\tau_-^{m-1}(\mathcal{L}_n)}$$

for all  $m \geq 1$  and  $n \geq -1$ .

**Proof:** Using Lemma 3.11, we can prove inductively that

$$\tau_-^m(\mathcal{L}_{n+1}) = R(\tau_-^m(\mathcal{L}_n)) + m \tau_-^{m-1}(\mathcal{L}_n) + \mathcal{G} * \overline{\tau_-^{m-1}(\mathcal{L}_n)}$$

for all  $m \geq 1$  and  $n \geq -1$ . The theorem then follows from Theorem 3.9.  $\square$

Recursively applying Theorem 4.8 and Lemma 4.1, we can express  $\overline{\tau_-^m(\mathcal{L}_n)}$  in terms of twisted quantum powers of  $\mathcal{X}$  for any  $m \geq 0$ . Here the twisting is given by the operation  $\mathcal{G}*$ . Such twisting is actually very important since it forces the sequence of vector fields  $\{\overline{\tau_-^m(\mathcal{L}_n)} \mid n \geq -1\}$  obey different linear relations for different  $m$ . This would make the span of vector fields  $\{\sum_{m=0}^k T^m(\overline{\tau_-^m(\mathcal{L}_n)}) \mid n \geq -1\}$  large enough for each fixed  $k$ , and consequently make the Virasoro conjecture more interesting. We omit the explicit formulas for  $\overline{\tau_-^m(\mathcal{L}_n)}$  for  $m > 1$  as it is not needed in this paper. To study the genus-2 Virasoro conjecture, we need the formula for  $m = 1$ . In this case Theorem 4.8 have the following form

$$\overline{\tau_-^1(\mathcal{L}_{n+1})} = \mathcal{X} \bullet \overline{\tau_-^1(\mathcal{L}_n)} - \overline{\mathcal{X}}^{n+1} - \mathcal{G} * \overline{\mathcal{X}}^{n+1}$$

for  $n \geq -1$ . Recursively applying this formula, we obtain

**Corollary 4.9**

$$\overline{\tau_-^1(\mathcal{L}_n)} = -\overline{\mathcal{X}}^{n+1} \bullet \tau_-(\mathcal{S}) - (n+1) \overline{\mathcal{X}}^n - \sum_{j=0}^n \overline{\mathcal{X}}^j \bullet (\mathcal{G} * \overline{\mathcal{X}}^{n-j})$$

for all  $n \geq 0$ .

Note that for  $m \geq 1$ ,  $\tau_-^m(\mathcal{S})$  is zero when restricted to the small phase. Therefore, on the small phase space, the first term on the right hand side of this formula will disappear.

It will be useful later to have a formula for covariant derivatives of  $\overline{\tau_-^1(\mathcal{L}_k)}$ .

**Corollary 4.10**

$$\nabla_{T(\mathcal{W})} \overline{\tau_-^1(\mathcal{L}_k)} = \overline{\mathcal{X}}^{k+1} \bullet \tau_-(\mathcal{W}) + (k+1) \overline{\mathcal{X}}^k \bullet \mathcal{W} + \sum_{i=0}^k \mathcal{X}^i \bullet (\mathcal{G} * (\overline{\mathcal{X}}^{k-i} \bullet \mathcal{W}))$$

for any vector field  $\mathcal{W}$  and  $k \geq -1$ .

**Proof:** By Lemma 3.4, Corollary 1.7, and equation (37),

$$\nabla_{T(\mathcal{W})} \{\mathcal{X}^k \bullet (\mathcal{G} * \overline{\mathcal{X}}^m)\} = -\mathcal{X}^k \bullet (\mathcal{G} * (\mathcal{W} \bullet \overline{\mathcal{X}}^m))$$

for any vector field  $\mathcal{W}$ . The corollary then follows from Corollary 4.9 and Corollary 3.6.  $\square$



## 5 Applications to the Virasoro conjecture

### 5.1 The Virasoro Conjecture

We will not describe the original version of the Virasoro conjecture as given in [EHX]. Instead we will use the following formulation of this conjecture

$$\langle\langle \mathcal{L}_n \rangle\rangle_g = \rho_{g,n}$$

for  $g \geq 0$  and  $n \geq -1$ . One of the advantages of this formulation is that the sequence of vector fields  $\mathcal{L}_n$  has a simple recursive definition (see equation (41)). The  $L_{-1}$ -constraint is just the string equation.  $L_0$ -constraint was discovered by Hori and is a combination of the quasi-homogeneity equation and the dilaton equation. Therefore for  $n = -1$  and 0, it is easy to figure out  $\rho_{g,n}$  from these equations. For the  $L_1$ -constraint, we have

$$\rho_{0,1} = \sum_{\alpha} \frac{1}{2} b_{\alpha} (1 - b_{\alpha}) \langle\langle \gamma_{\alpha} \rangle\rangle_0 \langle\langle \gamma^{\alpha} \rangle\rangle_0 - \sum_{\alpha, \beta} \frac{1}{2} (\mathcal{C}^2)_{\alpha\beta} t_0^{\alpha} t_0^{\beta},$$

and for genus  $g \geq 1$

$$\rho_{g,1} = - \sum_{\alpha} \frac{1}{2} b_{\alpha} (1 - b_{\alpha}) \left\{ \langle\langle \gamma_{\alpha} \gamma^{\alpha} \rangle\rangle_{g-1} + \sum_{h=1}^{g-1} \langle\langle \gamma_{\alpha} \rangle\rangle_h \langle\langle \gamma^{\alpha} \rangle\rangle_{g-h} \right\}.$$

For the  $L_2$ -constraint, we have

$$\begin{aligned} \rho_{0,2} &= \sum_{\alpha} b_{\alpha} (1 - b_{\alpha}^2) \langle\langle \tau_1(\gamma_{\alpha}) \rangle\rangle_0 \langle\langle \gamma^{\alpha} \rangle\rangle_0 \\ &\quad + \sum_{\alpha, \beta} \frac{1}{2} (1 - 3b_{\alpha}^2) \mathcal{C}_{\alpha}^{\beta} \langle\langle \gamma_{\beta} \rangle\rangle_0 \langle\langle \gamma^{\alpha} \rangle\rangle_0 - \frac{1}{2} (\mathcal{C}^3)_{\alpha\beta} t_0^{\alpha} t_0^{\beta}, \end{aligned}$$

and for genus  $g \geq 1$

$$\begin{aligned} \rho_{g,2} &= - \sum_{\alpha} b_{\alpha} (1 - b_{\alpha}^2) \left\{ \langle\langle \tau_1(\gamma_{\alpha}) \gamma^{\alpha} \rangle\rangle_{g-1} + \sum_{h=1}^{g-1} \langle\langle \tau_1(\gamma_{\alpha}) \rangle\rangle_h \langle\langle \gamma^{\alpha} \rangle\rangle_{g-h} \right\} \\ &\quad + \sum_{\alpha, \beta} \frac{1}{2} (1 - 3b_{\alpha}^2) \mathcal{C}_{\alpha}^{\beta} \left\{ \langle\langle \gamma_{\beta} \gamma^{\alpha} \rangle\rangle_{g-1} + \sum_{h=1}^{g-1} \langle\langle \gamma_{\beta} \rangle\rangle_h \langle\langle \gamma^{\alpha} \rangle\rangle_{g-h} \right\}. \end{aligned}$$

Since  $L_k$ -constraint is generated by  $L_1$  and  $L_2$  constraints for  $k \geq 1$ , this information suffices to determine the entire Virasoro conjecture.

It may be desirable to have a description of  $\rho_{g,n}$  in terms of recursive operators. For this purpose, we define

$$R_+(\mathcal{W}) := \mathcal{G} * \tau_+(\mathcal{W}) + C(\mathcal{W})$$

for any vector field  $\mathcal{W}$ . Then for  $n \geq 1$ ,

$$\begin{aligned} \rho_{0,n} &= \frac{1}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} \langle\langle \{ R_+^j(\mathcal{G} * C^i(\gamma_{\alpha})) \} \rangle\rangle_0 \langle\langle \{ R_+^{n-1-i-j}(\mathcal{G} * \gamma^{\alpha}) \} \rangle\rangle_0 \\ &\quad - \frac{1}{2} (\mathcal{C}^{n+1})_{\alpha\beta} t_0^{\alpha} t_0^{\beta} \end{aligned}$$

and for  $g \geq 1$ ,

$$\begin{aligned} \rho_{g,n} = & -\frac{1}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} \left\{ \left\langle \left\langle \left\{ R_+^j(\mathcal{G} * C^i(\gamma_\alpha)) \right\} \left\{ R_+^{n-1-i-j}(\mathcal{G} * \gamma^\alpha) \right\} \right\rangle \right\rangle_{g-1} \right. \\ & \left. + \sum_{h=1}^{g-1} \left\langle \left\langle \left\{ R_+^j(\mathcal{G} * C^i(\gamma_\alpha)) \right\} \right\rangle \right\rangle_h \left\langle \left\langle \left\{ R_+^{n-1-i-j}(\mathcal{G} * \gamma^\alpha) \right\} \right\rangle \right\rangle_{g-h} \right\}. \end{aligned}$$

Note that for  $g > 0$ ,  $\rho_{g,n}$  depends only on the data with genus less than  $g$ . Therefore the genus- $g$  Virasoro conjecture just computes  $\langle\langle \mathcal{L}_n \rangle\rangle_g$  in terms of data with genus less than  $g$ .

One might also formulate the following *weak Virasoro conjecture*:  $\langle\langle \mathcal{L}_n \rangle\rangle_g$  can be explicitly expressed in terms of data with genus less than  $g$  for  $g \geq 1$  and  $n \geq -1$ . This weak version of the Virasoro conjecture makes sense for all compact symplectic manifolds while the original version requires non-trivial topological conditions. From the computational point of view, once the weak Virasoro conjecture is proved, it will have the same computational power as the original Virasoro conjecture. This weak Virasoro conjecture can be generalized to the following *weak  $W$ -type constraints*:  $\langle\langle \tau_-^m(\mathcal{L}_n) \rangle\rangle_g$  can be explicitly expressed in terms of data with genus less than  $g$  for  $g \geq 1$ ,  $n \geq -1$  and  $m \geq 0$ . The computation of  $\overline{\tau_-^m(\mathcal{L}_n)}$  in terms of twisted quantum powers of the Euler vector field in Section 4 might be thought of as the genus-0 part of the  $W$ -constraints. While the work in [DZ2] implies the genus-1 part and results in this paper imply the genus-2 part of the weak  $W$ -constraints for manifolds with semisimple quantum cohomology in a rather trivial way, the  $W$ -type constraints would be interesting only when the Virasoro conjecture can not completely determine the generating functions.

The relationship between the quantum powers of the Euler vector field and the Virasoro vector fields as revealed in Lemma 4.1 may be thought of as an interpretation of the second derivatives of the genus-0 Virasoro conjecture. Vector fields  $\mathcal{X}^{k+1}$  are also closely related to the genus-1 Virasoro conjecture. In fact, equation (6) and (13) implies

$$\begin{aligned} \langle\langle \mathcal{L}_k \rangle\rangle_1 &= \langle\langle T(\tau_-(\mathcal{L}_k)) \rangle\rangle_1 + \langle\langle \overline{L}_k \rangle\rangle_1 \\ &= \frac{1}{24} \langle\langle \tau_-(\mathcal{L}_k) \gamma^\alpha \gamma_\alpha \rangle\rangle_0 - \langle\langle \overline{\mathcal{X}}^{k+1} \rangle\rangle_1. \end{aligned}$$

Therefore the genus-1  $L_k$ -constraint can be written as

$$\langle\langle \overline{\mathcal{X}}^{k+1} \rangle\rangle_1 = \frac{1}{24} \langle\langle \tau_-(\mathcal{L}_k) \gamma^\alpha \gamma_\alpha \rangle\rangle_0 - \rho_{1,k}.$$

It can be shown by using the genus-0 topological recursion relation and Lemma 4.1 that

$$\rho_{1,k} = -\frac{k+1}{8} \langle\langle \mathcal{X}^k \gamma_\alpha \gamma^\alpha \rangle\rangle_0 + \frac{1}{4} \sum_{i=0}^k b_\alpha b_\beta \langle\langle \gamma_\alpha \mathcal{X}^i \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \mathcal{X}^{k-i} \gamma^\alpha \rangle\rangle_0.$$

Together with Corollary 4.9, this explains the mysterious formulas in [L1] for  $\langle\langle \overline{\mathcal{X}}^{k+1} \rangle\rangle_1$  on the small phase space.

Since the genus-1 topological recursion relation is very powerful, the genus-1 Virasoro conjecture can be studied just using the quantum product on the small phase space. However, it seems that it is necessary to use the quantum product on the big phase space to study the genus-2 Virasoro conjecture since the genus-2 topological recursion relations are not strong enough. In fact, by equation (6),

$$\mathcal{L}_k = \overline{\mathcal{L}}_k + T(\overline{\tau_-(\mathcal{L}_k)}) + T^2(\tau_-^2(\mathcal{L}_k)).$$

So by the genus-2 topological recursion relation (19),

$$\langle\langle \mathcal{L}_k \rangle\rangle_2 = -\langle\langle \overline{\mathcal{X}}^{k+1} \rangle\rangle_2 + \langle\langle T(\overline{\tau_-(\mathcal{L}_k)}) \rangle\rangle_2 + A_1(\tau_-^2(\mathcal{L}_k)). \quad (46)$$

The second term on the right hand side contains descendant vector fields which seems can not be reduced to primary vector fields by the known topological recursion relations. It is expected that the situation would become worse at higher genera. Therefore we believe that the study of structures on the quantum product on the big phase space is essential in the study of the higher genera Virasoro conjecture.

Due to Corollary 4.9, the second term on the right hand side of equation (46) can be expressed in terms of quantum powers of the Euler vector field. This is also true for the last term on the right hand side of equation (46) because of Theorem 4.8 and the following formula

$$\begin{aligned} A_1(\mathcal{W}) &= A_1(\overline{\mathcal{W}}) + \frac{1}{20} \langle\langle \{\overline{\tau_-(\mathcal{W})} \bullet \gamma_\alpha \bullet \gamma^\alpha\} \rangle\rangle_1 \\ &\quad + \frac{1}{480} \langle\langle \{\overline{\tau_-(\mathcal{W})} \bullet \gamma_\alpha\} \gamma^\alpha \gamma_\beta \gamma^\beta \rangle\rangle_0 + \frac{1}{1152} \langle\langle \overline{\tau_-(\mathcal{W})} \{\gamma_\alpha \bullet \gamma^\alpha\} \gamma_\beta \gamma^\beta \rangle\rangle_0 \\ &\quad + \frac{1}{1152} \langle\langle \{\overline{\tau_-^2(\mathcal{W})} \bullet \gamma_\alpha \bullet \gamma^\alpha\} \gamma_\beta \gamma^\beta \rangle\rangle_0. \end{aligned} \quad (47)$$

This equation follows from the definition of  $A_1$ , Corollary 1.6, equations (9) and (14).

In the rest of this section we will study the genus-2 Virasoro conjecture. We also solve the genus-2 generating function in terms of genus-0 and genus-1 data for manifolds whose quantum cohomology is not too degenerate.

## 5.2 Reduce the genus-2 Virasoro conjecture to the genus-2 $L_1$ -constraint

In this subsection, we prove that the genus-2 Virasoro conjecture can be reduced to the  $L_1$ -constraint provided that the genus-1  $L_1$ -constraint is satisfied. Note that the genus-1  $L_1$ -constraint holds if and only if the genus-1 Virasoro conjecture holds (cf. [L1]). For a discussion of some sufficient conditions for the genus-1 Virasoro conjecture, see [L1] and [L2].

Define

$$\psi_k := \langle\langle \overline{\mathcal{X}}^k \rangle\rangle_2 - \langle\langle T(\overline{\tau_-(\mathcal{L}_{k-1})}) \rangle\rangle_2. \quad (48)$$

Then genus-2  $L_k$ -constraint have the following form

$$\psi_{k+1} = A_1(\tau_-^2(\mathcal{L}_k)) - \rho_{2,k}.$$

We first apply equation (21) to the case  $W_i = \mathcal{X}$  and obtain

$$\begin{aligned} B(\mathcal{X}, \mathcal{X}, \mathcal{X}) &= 2 \langle\langle \mathcal{X}^3 \rangle\rangle_2 - 2 \langle\langle \mathcal{X} \mathcal{X} \mathcal{X} \gamma^\alpha \rangle\rangle_0 \langle\langle T(\gamma_\alpha) \rangle\rangle_2 \\ &\quad - 3 \langle\langle T(\mathcal{X}) \mathcal{X}^2 \rangle\rangle_2 + 3 \langle\langle \mathcal{X} T(\mathcal{X}^2) \rangle\rangle_2. \end{aligned}$$

Since

$$\langle\langle T(\mathcal{X}) \mathcal{X}^2 \rangle\rangle_2 = T(\mathcal{X}) \langle\langle \mathcal{X}^2 \rangle\rangle_2 - \langle\langle \{ \nabla_{T(\mathcal{X})} \mathcal{X}^2 \} \rangle\rangle_2 = T(\mathcal{X}) \langle\langle \mathcal{X}^2 \rangle\rangle_2 + \langle\langle \mathcal{X}^3 \rangle\rangle_2$$

and

$$\begin{aligned} \langle\langle \mathcal{X} T(\mathcal{X}^2) \rangle\rangle_2 &= T(\mathcal{X}^2) \langle\langle \mathcal{X} \rangle\rangle_2 - \langle\langle \{ \nabla_{T(\mathcal{X}^2)} \mathcal{X} \} \rangle\rangle_2 \\ &= -(3b_1 + 2) \langle\langle T(\mathcal{X}^2) \rangle\rangle_2 + \langle\langle T(\mathcal{G} * \mathcal{X}^2) \rangle\rangle_2 + \langle\langle \mathcal{X}^3 \rangle\rangle_2, \end{aligned}$$

we have

$$B(\mathcal{X}, \mathcal{X}, \mathcal{X}) = 2 \langle\langle \mathcal{X}^3 \rangle\rangle_2 - 3T(\mathcal{X}) \langle\langle \mathcal{X}^2 \rangle\rangle_2 - \langle\langle T(\mathcal{V}_1) \rangle\rangle_2 \quad (49)$$

where

$$\mathcal{V}_1 = 2 \langle\langle \mathcal{X} \mathcal{X} \mathcal{X} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha + 3(3b_1 + 2) \mathcal{X}^2 - 3 \mathcal{G} * \mathcal{X}^2.$$

Using this formula, we can prove the following

**Lemma 5.1** *If the genus-2  $L_1$ -constraint holds, then*

$$\psi_3 = \frac{3}{2} \left\{ A_1(\nabla_{T(\mathcal{X})} \tau_-^2(\mathcal{L}_1)) + A_2(\mathcal{X}, \tau_-(\mathcal{L}_1)) \right\} + \frac{1}{2} B(\mathcal{X}, \mathcal{X}, \mathcal{X}) - \frac{3}{2} T(\mathcal{X}) \rho_{2,1}.$$

**Proof:** We first observe that by Lemma 3.14, Corollaries 4.9 and 4.10, and equation (36), the vector field  $\mathcal{V}_1$  defined after equation (49) also satisfies

$$\mathcal{V}_1 = 2 \overline{\tau_-(\mathcal{L}_2)} - 9 \mathcal{X} \bullet \overline{\tau_-(\mathcal{L}_1)} - 3 \nabla_{T(\mathcal{X})} \overline{\tau_-(\mathcal{L}_1)}.$$

Secondly, by equation (5),

$$T(\tau_-(\mathcal{W})) = \tau_+(\tau_-(\mathcal{W})) - \overline{\tau_+(\tau_-(\mathcal{W}))} = \mathcal{W} - \overline{\mathcal{W}}$$

for any vector field  $\mathcal{W}$ . Therefore by Corollary 2.3,

$$\begin{aligned} T(\mathcal{X}) A_1(\tau_-^2(\mathcal{L}_1)) &= A_2(\mathcal{X}, T(\tau_-^2(\mathcal{L}_1))) + A_1(\nabla_{T(\mathcal{X})} \tau_-^2(\mathcal{L}_1)) \\ &= A_2(\mathcal{X}, \tau_-(\mathcal{L}_1) - \overline{\tau_-(\mathcal{L}_1)}) + A_1(\nabla_{T(\mathcal{X})} \tau_-^2(\mathcal{L}_1)). \end{aligned}$$

Moreover, by equation (20),

$$\begin{aligned} T(\mathcal{X}) \left\langle\left\langle T(\overline{\tau_-(\mathcal{L}_1)}) \right\rangle\right\rangle_2 &= \left\langle\left\langle T(\mathcal{X}) T(\overline{\tau_-(\mathcal{L}_1)}) \right\rangle\right\rangle_2 + \left\langle\left\langle \left\{ \nabla_{T(\mathcal{X})} T(\overline{\tau_-(\mathcal{L}_1)}) \right\} \right\rangle\right\rangle_2 \\ &= A_2(\mathcal{X}, \overline{\tau_-(\mathcal{L}_1)}) + 3 \left\langle\left\langle T(\mathcal{X} \bullet \overline{\tau_-(\mathcal{L}_1)}) \right\rangle\right\rangle_2 + \left\langle\left\langle T(\nabla_{T(\mathcal{X})} \overline{\tau_-(\mathcal{L}_1)}) \right\rangle\right\rangle_2. \end{aligned}$$

If genus-2  $L_1$ -constraint is satisfied, then

$$\left\langle\left\langle \mathcal{X}^2 \right\rangle\right\rangle_2 = \left\langle\left\langle T(\overline{\tau_-(\mathcal{L}_1)}) \right\rangle\right\rangle_2 + A_1(\tau_-^2(\mathcal{L}_1)) - \rho_{2,1}.$$

The lemma then follows by plugging this equation into equation (49).  $\square$

### Lemma 5.2

$$\nabla_{T(\mathcal{X})} \tau_-^2(\mathcal{L}_1) = -\tau_-^2(\mathcal{L}_2) + (b_1 + 1)\tau_-(\mathcal{L}_1) + 2(b_1 + 1)\mathcal{L}_0 - 2(b_1 + 1)\mathcal{D}.$$

**Proof:** By Lemma 4.5 and the fact that covariant derivatives commute with  $\tau_-$ ,

$$\nabla_{T(\mathcal{X})} \tau_-^2(\mathcal{L}_1) = \tau_-^2 R^2(\mathcal{X}) = \tau_-^2 R^2(-\mathcal{L}_0 - (b_1 + 1)\mathcal{D}) = -\tau_-^2(\mathcal{L}_2) - (b_1 + 1)\tau_-^2 R^2(\mathcal{D}).$$

Since  $\overline{\mathcal{D}} = 0$ , by Lemma 3.11,

$$\tau_- R(\mathcal{D}) = R\tau_-(\mathcal{D}) + \mathcal{D} = R(\mathcal{S}) + \mathcal{D} = -\mathcal{L}_0 + \mathcal{D}.$$

So

$$\tau_- R\tau_- R(\mathcal{D}) = -\tau_-(\mathcal{L}_1) - \mathcal{L}_0 + \mathcal{D}$$

and

$$\tau_-^2 R^2(\mathcal{D}) = \tau_- R\tau_- R(\mathcal{D}) + \tau_-(\mathcal{G} * \overline{R(\mathcal{D})}) + \tau_- R(\mathcal{D}) = -\tau_-(\mathcal{L}_1) - 2\mathcal{L}_0 + 2\mathcal{D}.$$

The lemma follows.  $\square$

### Lemma 5.3

$$A_1(\mathcal{D}) = \frac{1}{20} \left\langle\left\langle \{\gamma^\alpha \bullet \gamma_\alpha\} \right\rangle\right\rangle_1 + \frac{1}{480} \left\langle\left\langle \gamma_\alpha \gamma^\alpha \gamma_\beta \gamma^\beta \right\rangle\right\rangle_0,$$

and

$$\begin{aligned} A_1(\mathcal{L}_0) &= -\frac{7}{10} \left\langle\left\langle \{\mathcal{X} \bullet \gamma^\alpha\} \right\rangle\right\rangle_1 \left\langle\left\langle \gamma_\alpha \right\rangle\right\rangle_1 - \frac{1}{10} \left\langle\left\langle \{\mathcal{X} \bullet \gamma^\alpha\} \gamma_\alpha \right\rangle\right\rangle_1 \\ &\quad + \frac{1}{120} (7b_\beta - 13) \left\langle\left\langle \gamma^\alpha \gamma_\alpha \gamma^\beta \right\rangle\right\rangle_0 \left\langle\left\langle \gamma_\beta \right\rangle\right\rangle_1 \\ &\quad - \frac{1}{480} \left\langle\left\langle \gamma_\alpha \gamma^\alpha \gamma_\beta \gamma^\beta \right\rangle\right\rangle_0. \end{aligned}$$

**Proof:** The formula for  $A_1(\mathcal{D})$  follows from derivatives of the dilaton equation

$$\langle\langle \mathcal{D} \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g = (k + 2g - 2) \langle\langle \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g$$

for any vector fields  $\mathcal{W}_1, \dots, \mathcal{W}_k$ . The formula for  $A_1(\mathcal{L}_0)$  is obtained by applying derivatives of the  $L_0$ -constraint

$$\begin{aligned} \langle\langle \mathcal{L}_0 \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g &= - \sum_{i=1}^k \langle\langle \mathcal{W}_1 \cdots \{\tau_- R(\mathcal{W}_i)\} \cdots \mathcal{W}_k \rangle\rangle_g \\ &\quad + k \langle\langle \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g - \delta_{g,0} \nabla_{\mathcal{W}_1, \dots, \mathcal{W}_k}^k \left( \frac{1}{2} \mathcal{C}_{\alpha\beta} t_0^\alpha t_0^\beta \right) \end{aligned}$$

to remove  $\mathcal{L}_0$  from genus-0 correlation functions with more than 3 points and genus-1 correlation functions.  $\square$

To compute  $B(\mathcal{X}, \mathcal{X}, \mathcal{X})$ , we need the following

**Lemma 5.4** *If genus-1  $L_1$ -constraint holds, then*

$$\begin{aligned} \langle\langle \mathcal{X}^2 \gamma^\alpha \rangle\rangle_1 &= (1 - b_\alpha + b_\beta) \langle\langle \mathcal{X} \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 \\ &\quad + \left\{ \frac{1}{2} b_\beta (1 - b_\beta) + \frac{1}{24} (1 - b_\alpha) (2 - b_\alpha) \right\} \langle\langle \gamma^\alpha \gamma_\beta \gamma^\beta \rangle\rangle_0 \\ &\quad + \langle\langle \{\tau_-(\mathcal{L}_1) \bullet \gamma^\alpha\} \rangle\rangle_1 + \frac{1}{24} \langle\langle \tau_-(\mathcal{L}_1) \gamma^\alpha \gamma_\beta \gamma^\beta \rangle\rangle_0 \end{aligned}$$

and

$$\begin{aligned} \langle\langle \mathcal{X}^2 \gamma^\alpha \gamma_\alpha \rangle\rangle_1 &= 2(1 - b_\alpha + b_\beta) \langle\langle \mathcal{X} \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_1 \\ &\quad + \{(1 - b_\alpha + b_\beta)(2 - b_\alpha - b_\beta) + b_\alpha(b_\alpha + 1)\} \langle\langle \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 \\ &\quad + \left\{ \frac{1}{2} b_\beta (1 - b_\beta) + \frac{1}{24} (1 - b_\alpha) (2 - b_\alpha) + \frac{1}{24} b_\alpha (b_\alpha + 1) \right\} \langle\langle \gamma_\alpha \gamma^\alpha \gamma_\beta \gamma^\beta \rangle\rangle_0 \\ &\quad + 2 \langle\langle \{\tau_-(\mathcal{L}_1) \bullet \gamma^\alpha\} \gamma_\alpha \rangle\rangle_1 + \langle\langle \tau_-(\mathcal{L}_1) \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 \\ &\quad + \frac{1}{24} \langle\langle \tau_-(\mathcal{L}_1) \gamma_\alpha \gamma^\alpha \gamma_\beta \gamma^\beta \rangle\rangle_0. \end{aligned}$$

**Proof:** First observe that

$$\mathcal{X}^2 = -\overline{\mathcal{L}}_1 = -\mathcal{L}_1 + T(\tau_-(\mathcal{L}_1))$$

and

$$\begin{aligned} \nabla_{\gamma_\mu} \mathcal{L}_1 &= \nabla_{\gamma_\mu} R(\mathcal{L}_0) = R(\nabla_{\gamma_\mu} \mathcal{L}_0) - \mathcal{G} * (\mathcal{L}_0 \bullet \gamma_\mu) \\ &= \mathcal{X} \bullet (b_\mu \gamma_\mu) + T(\tau_- R(b_\mu \gamma_\mu)) + \mathcal{G} * (\mathcal{X} \bullet \gamma_\mu) \\ &= (b_\mu + b_\alpha) \langle\langle \mathcal{X} \gamma_\mu \gamma^\alpha \rangle\rangle_0 \gamma_\alpha + b_\mu (b_\mu + 1) T(\gamma_\mu). \end{aligned}$$

The formulas in the lemma are obtained by applying equations (14) and (15) and taking derivatives of the genus-1  $L_1$ -constraint which has the following form

$$\langle\langle \mathcal{L}_1 \rangle\rangle_1 = - \sum_\alpha \frac{1}{2} b_\alpha (1 - b_\alpha) \langle\langle \gamma_\alpha \gamma^\alpha \rangle\rangle_0.$$

$\square$

**Lemma 5.5** *If genus-1  $L_1$ -constraint holds, then*

$$\begin{aligned}
& 3A_2(\mathcal{X}, \tau_-(\mathcal{L}_1)) + B(\mathcal{X}, \mathcal{X}, \mathcal{X}) \\
= & 5A_1(\tau_-^2(\mathcal{L}_2)) - 3(b_1 + 1)A_1(\tau_-(\mathcal{L}_1)) \\
& + \frac{1}{5} \{5b_\alpha(b_\alpha + b_\beta) - 5b_\alpha + 21(b_1 + 1)\} \langle\langle \mathcal{X} \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_1 \langle\langle \gamma_\beta \rangle\rangle_1 \\
& + \frac{1}{10} \{10b_\alpha(b_\alpha + b_\beta) - 10b_\alpha + 6(b_1 + 1)\} \langle\langle \mathcal{X} \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_1 \\
& + \frac{1}{120} \left\{ -5b_\beta^3 - 15b_1b_\beta^2 - 27b_1b_\beta - 37b_\beta + 114(b_1 + 1) \right. \\
& \quad \left. + 180(b_\beta + b_1 + 1)b_\alpha(1 - b_\alpha) \right\} \langle\langle \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 \\
& + \frac{1}{40} (b_1 + 1)(-5b_\alpha^2 + 5b_\alpha + 1) \langle\langle \gamma_\alpha \gamma^\alpha \gamma_\beta \gamma^\beta \rangle\rangle_0.
\end{aligned}$$

**Proof:** We first observe that

$$\begin{aligned}
\tau_- R(\mathcal{X}) &= \tau_- R(-\mathcal{L}_0 - (b_1 + 1)\mathcal{D}) = -\tau_-(\mathcal{L}_1) - (b_1 + 1)\tau_- R(\mathcal{D}) \\
&= -\tau_-(\mathcal{L}_1) + (b_1 + 1)\mathcal{L}_0 - (b_1 + 1)\mathcal{D}.
\end{aligned}$$

Similarly, we have

$$\tau_- R \tau_- R(\mathcal{X}) = -\tau_-^2(\mathcal{L}_2) + (b_1 + 2)\tau_-(\mathcal{L}_1) + (b_1 + 1)\mathcal{L}_0 - (b_1 + 1)\mathcal{D}$$

and

$$\tau_- R(\tau_-(\mathcal{L}_1)) = \tau_-^2(\mathcal{L}_2) - \tau_-(\mathcal{L}_1).$$

Remove  $\mathcal{X}$  from genus-0 correlation functions with more than 3 points and genus-1 correlation functions in  $A_2(\mathcal{X}, \tau_-(\mathcal{L}_1))$  and  $B(\mathcal{X}, \mathcal{X}, \mathcal{X})$  using Lemma 3.13 and the above formulas. Then use Lemma 5.4 to replace  $\langle\langle \mathcal{X}^2 \gamma^\alpha \rangle\rangle_1$  and  $\langle\langle \mathcal{X}^2 \gamma^\alpha \gamma_\alpha \rangle\rangle_1$ . The lemma then follows from simplifying the resulting expression.  $\square$

The last term of the formula in Lemma 5.1 can be computed in the following way

**Lemma 5.6**

$$\begin{aligned}
-T(\mathcal{X})\rho_{2,1} &= b_\alpha(1 - b_\alpha) \{ \langle\langle \{\mathcal{X} \bullet \gamma^\alpha\} \rangle\rangle_1 \langle\langle \gamma_\alpha \rangle\rangle_1 + \langle\langle \{\mathcal{X} \bullet \gamma^\alpha\} \gamma_\alpha \rangle\rangle_1 \} \\
&+ \frac{1}{2} \left\{ b_\alpha(1 - b_\alpha) + \frac{1}{12} b_\beta(1 - b_\beta) \right\} (1 - b_1 - b_\beta) \langle\langle \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 \\
&- \frac{1}{24} b_1 b_\alpha(1 - b_\alpha) \langle\langle \gamma_\alpha \gamma^\alpha \gamma_\beta \gamma^\beta \rangle\rangle_0.
\end{aligned}$$

**Proof:** This formula is obtained by applying equations (14) and (15) first, then remove  $\mathcal{X}$  from genus-0 correlation functions with more than 3 points.  $\square$

To write the prediction of the genus-2  $L_2$  constraint in a form consistent with the above calculations, we need the following

**Lemma 5.7**

$$\sum_{\alpha} b_{\alpha}(1 - b_{\alpha}^2) \langle\langle \gamma_{\alpha} \gamma^{\alpha} \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g = \frac{3}{2} \sum_{\alpha} b_{\alpha}(1 - b_{\alpha}) \langle\langle \gamma_{\alpha} \gamma^{\alpha} \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g$$

for any vector fields  $\mathcal{W}_1, \dots, \mathcal{W}_k$ .

**Proof:** The difference of the right hand side and the left hand side of this equation is

$$\begin{aligned} & \frac{1}{2} \sum_{\alpha} b_{\alpha}(1 - b_{\alpha})(1 - 2b_{\alpha}) \langle\langle \gamma_{\alpha} \gamma^{\alpha} \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g \\ &= \frac{1}{2} \sum_{\alpha, \beta} b_{\alpha}(1 - b_{\alpha})(1 - 2b_{\alpha}) \eta^{\alpha\beta} \langle\langle \gamma_{\alpha} \gamma_{\beta} \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g \\ &= \frac{1}{2} \sum_{\alpha, \beta} (1 - b_{\beta}) b_{\beta} (-1 + 2b_{\beta}) \eta^{\alpha\beta} \langle\langle \gamma_{\alpha} \gamma_{\beta} \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g \\ &= -\frac{1}{2} \sum_{\beta} b_{\beta}(1 - b_{\beta})(1 - 2b_{\beta}) \langle\langle \gamma^{\beta} \gamma_{\beta} \mathcal{W}_1 \cdots \mathcal{W}_k \rangle\rangle_g. \end{aligned}$$

Here we have used the fact that  $b_{\alpha} + b_{\beta} = 1$  if  $\eta^{\alpha\beta} \neq 0$ . Comparing the two sides of this equation, both of them must be zero.  $\square$

Now we can rewrite the prediction of  $-\langle\langle \mathcal{L}_2 \rangle\rangle_2$  given by the Virasoro conjecture in the following form

**Lemma 5.8**

$$\begin{aligned} -\rho_{2,2} &= \frac{1}{2} \left\{ -2b_{\alpha}^2 + 2b_{\alpha} + b_{\alpha}b_{\beta} \right\} \langle\langle \mathcal{X} \gamma^{\alpha} \gamma^{\beta} \rangle\rangle_0 \left\{ \langle\langle \gamma_{\alpha} \rangle\rangle_1 \langle\langle \gamma_{\beta} \rangle\rangle_1 + \langle\langle \gamma_{\alpha} \gamma_{\beta} \rangle\rangle_1 \right\} \\ &+ \left\{ \frac{3}{2} b_{\alpha}(1 - b_{\alpha}) + \frac{1}{24} b_{\beta}(1 - b_{\beta})(2 - b_{\beta}) \right\} \langle\langle \gamma^{\alpha} \gamma_{\alpha} \gamma^{\beta} \rangle\rangle_0 \langle\langle \gamma_{\beta} \rangle\rangle_1 \\ &+ \frac{1}{16} b_{\alpha}(1 - b_{\alpha}) \langle\langle \gamma_{\alpha} \gamma^{\alpha} \gamma_{\beta} \gamma^{\beta} \rangle\rangle_0. \end{aligned}$$

**Proof:** Applying equation (14), (15) and Lemma 5.7, we have

$$\begin{aligned} & \sum_{\alpha} b_{\alpha}(1 - b_{\alpha}^2) \left\{ \langle\langle \tau_1(\gamma_{\alpha}) \gamma^{\alpha} \rangle\rangle_1 + \langle\langle \tau_1(\gamma_{\alpha}) \rangle\rangle_1 \langle\langle \gamma^{\alpha} \rangle\rangle_1 \right\} \\ &= \sum_{\alpha, \beta} b_{\alpha}(1 - b_{\alpha}^2) \langle\langle \gamma_{\alpha} \gamma_{\beta} \rangle\rangle_0 \left\{ \langle\langle \gamma^{\alpha} \gamma^{\beta} \rangle\rangle_1 + \langle\langle \gamma^{\alpha} \rangle\rangle_1 \langle\langle \gamma^{\beta} \rangle\rangle_1 \right\} \\ &+ \left\{ \frac{3}{2} b_{\alpha}(1 - b_{\alpha}) + \frac{1}{24} b_{\beta}(1 - b_{\beta})(2 - b_{\beta}) \right\} \langle\langle \gamma^{\alpha} \gamma_{\alpha} \gamma^{\beta} \rangle\rangle_0 \langle\langle \gamma_{\beta} \rangle\rangle_1 \\ &+ \frac{1}{16} b_{\alpha}(1 - b_{\alpha}) \langle\langle \gamma_{\alpha} \gamma^{\alpha} \gamma_{\beta} \gamma^{\beta} \rangle\rangle_0. \end{aligned}$$

By the symmetry of  $\alpha$  and  $\beta$ , the first term on the right hand side can be written as

$$\begin{aligned} & \sum_{\alpha, \beta} \frac{1}{2} \left\{ b_{\alpha}(1 - b_{\alpha}^2) + b_{\beta}(1 - b_{\beta}^2) \right\} \langle\langle \gamma_{\alpha} \gamma_{\beta} \rangle\rangle_0 \left\{ \langle\langle \gamma^{\alpha} \gamma^{\beta} \rangle\rangle_1 + \langle\langle \gamma^{\alpha} \rangle\rangle_1 \langle\langle \gamma^{\beta} \rangle\rangle_1 \right\} \\ &= \sum_{\alpha, \beta} \frac{1}{2} (1 - b_{\alpha}^2 + b_{\alpha}b_{\beta} - b_{\beta}^2) \left\{ (b_{\alpha} + b_{\beta}) \langle\langle \gamma_{\alpha} \gamma_{\beta} \rangle\rangle_0 \right\} \left\{ \langle\langle \gamma^{\alpha} \gamma^{\beta} \rangle\rangle_1 + \langle\langle \gamma^{\alpha} \rangle\rangle_1 \langle\langle \gamma^{\beta} \rangle\rangle_1 \right\} \end{aligned}$$



By equation (29),

$$(b_\alpha + b_\beta) \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_0 = \langle\langle \mathcal{X} \gamma_\alpha \gamma_\beta \rangle\rangle_0 - \mathcal{C}_{\alpha\beta}.$$

Since  $\mathcal{C}_{\alpha\beta} \neq 0$  implies  $b_\alpha = -b_\beta$ ,

$$(1 - b_\alpha^2 + b_\alpha b_\beta - b_\beta^2) \mathcal{C}_{\alpha\beta} = (1 - 3b_\alpha^2) \mathcal{C}_{\alpha\beta}.$$

The lemma then follows from interchanging upper indices and lower indices and using the symmetry of  $\alpha$  and  $\beta$ .  $\square$

We can now prove

**Theorem 5.9** *For any manifold which satisfies the genus-1  $L_1$ -constraint, if the genus-2  $L_1$ -constraint holds, then the genus-2 Virasoro conjecture holds.*

**Proof:** Combining the results in Lemmas 5.1 - 5.3, 5.5 and 5.6, we obtain a formula for  $\psi_3$ . Due to Lemma 5.8, this formula coincides with the prediction of the genus-2  $L_2$ -constraint. Since  $L_k$ -constraint is generated by  $L_{-1}$  and  $L_2$ -constraints, the theorem is thus proved.  $\square$

### 5.3 A recursive formula for $\psi_k$

In this subsection, we prove a recursive formula for the function  $\psi_k$  which was defined in equation (48). We first apply equation (21) to  $\mathcal{W}_i = \overline{\mathcal{X}}^{m_i}$ , for  $i = 1, 2, 3$  and  $m_i \geq 0$  and obtain

$$\begin{aligned} & 2 \langle\langle \overline{\mathcal{X}}^m \rangle\rangle_2 - 2 \langle\langle \overline{\mathcal{X}}^{m_1} \overline{\mathcal{X}}^{m_2} \overline{\mathcal{X}}^{m_3} \gamma^\alpha \rangle\rangle_0 \langle\langle T(\gamma_\alpha) \rangle\rangle_2 \\ & + \sum_{i=1}^3 \left\{ \langle\langle \overline{\mathcal{X}}^{m_i} T(\overline{\mathcal{X}}^{m-m_i}) \rangle\rangle_2 - \langle\langle T(\overline{\mathcal{X}}^{m_i}) \overline{\mathcal{X}}^{m-m_i} \rangle\rangle_2 \right\} \\ & = B(\overline{\mathcal{X}}^{m_1}, \overline{\mathcal{X}}^{m_2}, \overline{\mathcal{X}}^{m_3}), \end{aligned} \tag{50}$$

where  $m = m_1 + m_2 + m_3$ .

To simplify this equation, we notice that, by Lemma 3.4,

$$\begin{aligned} \langle\langle \overline{\mathcal{X}}^n T(\overline{\mathcal{X}}^k) \rangle\rangle_2 &= T(\overline{\mathcal{X}}^k) \langle\langle \overline{\mathcal{X}}^n \rangle\rangle_2 - \langle\langle \{\nabla_{T(\overline{\mathcal{X}}^k)} \overline{\mathcal{X}}^n\} \rangle\rangle_2 \\ &= T(\overline{\mathcal{X}}^k) \langle\langle \overline{\mathcal{X}}^n \rangle\rangle_2 + \langle\langle \overline{\mathcal{X}}^{n+k} \rangle\rangle_2 \end{aligned}$$

for any  $n, k \geq 0$ . Hence equation (50) can be written as

$$\begin{aligned} & 2 \langle\langle \overline{\mathcal{X}}^m \rangle\rangle_2 - 2 \langle\langle \overline{\mathcal{X}}^{m_1} \overline{\mathcal{X}}^{m_2} \overline{\mathcal{X}}^{m_3} \gamma^\alpha \rangle\rangle_0 \langle\langle T(\gamma_\alpha) \rangle\rangle_2 \\ & + \sum_{i=1}^3 \left\{ T(\overline{\mathcal{X}}^{m-m_i}) \langle\langle \overline{\mathcal{X}}^{m_i} \rangle\rangle_2 - T(\overline{\mathcal{X}}^{m_i}) \langle\langle \overline{\mathcal{X}}^{m-m_i} \rangle\rangle_2 \right\} \\ & = B(\overline{\mathcal{X}}^{m_1}, \overline{\mathcal{X}}^{m_2}, \overline{\mathcal{X}}^{m_3}). \end{aligned} \tag{51}$$

To simplify this equation further, we need the following lemma.

**Lemma 5.10**

$$\begin{aligned}
& T(\overline{\mathcal{X}}^k) \langle\langle T(\overline{\tau_-(\mathcal{L}_{n-1})}) \rangle\rangle_2 - T(\overline{\mathcal{X}}^n) \langle\langle T(\overline{\tau_-(\mathcal{L}_{k-1})}) \rangle\rangle_2 \\
&= 2(k-n) \langle\langle T(\overline{\mathcal{X}^{k+n-1}}) \rangle\rangle_2 + 2 \sum_{i=0}^{k-1} \langle\langle T(\mathcal{X}^i \bullet (\mathcal{G} * \overline{\mathcal{X}^{k+n-1-i}})) \rangle\rangle_2 \\
&\quad - 2 \sum_{i=0}^{n-1} \langle\langle T(\mathcal{X}^i \bullet (\mathcal{G} * \overline{\mathcal{X}^{k+n-1-i}})) \rangle\rangle_2 \\
&\quad + A_2(\overline{\mathcal{X}^k}, \overline{\tau_-(\mathcal{L}_{n-1})}) - A_2(\overline{\mathcal{X}^n}, \overline{\tau_-(\mathcal{L}_{k-1})})
\end{aligned}$$

for  $k, n \geq 0$ .

**Proof:** By equation (7),

$$\begin{aligned}
& T(\overline{\mathcal{X}}^k) \langle\langle T(\overline{\tau_-(\mathcal{L}_{n-1})}) \rangle\rangle_2 \\
&= \langle\langle T(\overline{\mathcal{X}}^k) T(\overline{\tau_-(\mathcal{L}_{n-1})}) \rangle\rangle_2 + \langle\langle \left\{ \nabla_{T(\overline{\mathcal{X}}^k)} T(\overline{\tau_-(\mathcal{L}_{n-1})}) \right\} \rangle\rangle_2.
\end{aligned}$$

By equation (20) and Lemma 1.5,

$$\begin{aligned}
& T(\overline{\mathcal{X}}^k) \langle\langle T(\overline{\tau_-(\mathcal{L}_{n-1})}) \rangle\rangle_2 \\
&= 3 \langle\langle T(\mathcal{X}^k \bullet \overline{\tau_-(\mathcal{L}_{n-1})}) \rangle\rangle_2 + A_2(\overline{\mathcal{X}^k}, \overline{\tau_-(\mathcal{L}_{n-1})}) \\
&\quad + \langle\langle T(\nabla_{T(\overline{\mathcal{X}}^k)} \overline{\tau_-(\mathcal{L}_{n-1})}) \rangle\rangle_2.
\end{aligned}$$

The lemma then follows from Corollary 4.9 and Corollary 4.10.  $\square$

**Theorem 5.11** For  $m_1, m_2, m_3 \geq 0$  and  $m = m_1 + m_2 + m_3$ ,

$$\begin{aligned}
& 2\psi_m + \sum_{i=1}^3 \left\{ T(\overline{\mathcal{X}^{m-m_i}}) \psi_{m_i} - T(\overline{\mathcal{X}^{m_i}}) \psi_{m-m_i} \right\} \\
&= B(\overline{\mathcal{X}^{m_1}}, \overline{\mathcal{X}^{m_2}}, \overline{\mathcal{X}^{m_3}}) \\
&\quad + \sum_{i=1}^3 \left\{ A_2(\overline{\mathcal{X}^{m_i}}, \overline{\tau_-(\mathcal{L}_{m-m_i-1})}) - A_2(\overline{\mathcal{X}^{m-m_i}}, \overline{\tau_-(\mathcal{L}_{m_i-1})}) \right\}.
\end{aligned}$$

**Proof:** Plugging  $\langle\langle \overline{\mathcal{X}^k} \rangle\rangle_2 = \psi_k + \langle\langle T(\overline{\tau_-(\mathcal{L}_{k-1})}) \rangle\rangle_2$  into equation (51) and applying Lemma 5.10, we obtain

$$\begin{aligned}
& 2\psi_m + 2 \langle\langle T(\mathcal{Y}) \rangle\rangle_2 + \sum_{i=1}^3 \left\{ T(\overline{\mathcal{X}^{m-m_i}}) \psi_{m_i} - T(\overline{\mathcal{X}^{m_i}}) \psi_{m-m_i} \right\} \\
&\quad + \sum_{i=1}^3 \left\{ A_2(\overline{\mathcal{X}^{m-m_i}}, \overline{\tau_-(\mathcal{L}_{m_i-1})}) - A_2(\overline{\mathcal{X}^{m_i}}, \overline{\tau_-(\mathcal{L}_{m-m_i-1})}) \right\} \\
&= B(\overline{\mathcal{X}^{m_1}}, \overline{\mathcal{X}^{m_2}}, \overline{\mathcal{X}^{m_3}}).
\end{aligned}$$

where

$$\begin{aligned} \mathcal{Y} = & \overline{\tau_-(\mathcal{L}_{m-1})} - \langle\langle \overline{\mathcal{X}}^{m_1} \overline{\mathcal{X}}^{m_2} \overline{\mathcal{X}}^{m_3} \gamma^\alpha \rangle\rangle_2 \gamma_\alpha \\ & + \sum_{i=1}^3 \left\{ (m - 2m_i) \overline{\mathcal{X}}^{m-1} + \sum_{j=0}^{m-m_i-1} \mathcal{X}^j \bullet (\mathcal{G} * \overline{\mathcal{X}}^{m-1-j}) \right. \\ & \left. - \sum_{j=0}^{m_i-1} \mathcal{X}^j \bullet (\mathcal{G} * \overline{\mathcal{X}}^{m-1-j}) \right\}. \end{aligned}$$

By Corollary 3.17 and Corollary 4.9,

$$\mathcal{Y} = 0.$$

The theorem is thus proved.  $\square$

**Corollary 5.12** *For any manifold,*

$$\begin{aligned} & 2(k-1)\psi_{k+1} - (k+1)T(\overline{\mathcal{X}})\psi_k \\ = & (k+1)A_2(\overline{\mathcal{X}}, \overline{\tau_-(\mathcal{L}_{k-1})}) - (k+1)A_2(\overline{\mathcal{X}}^k, \overline{\tau_-(\mathcal{L}_0)}) - (k+1)T(\overline{\mathcal{X}}^k)A_1(\tau_-^2(\mathcal{L}_0)) \\ & - \delta_{k,0}B(\overline{\mathcal{X}}, \overline{\mathcal{S}}, \overline{\mathcal{S}}) + \sum_{j=1}^{k-1} B(\overline{\mathcal{X}}, \overline{\mathcal{X}}^j, \overline{\mathcal{X}}^{k-j}) \end{aligned}$$

for  $k \geq 0$ . Here the last summation should be understood as 0 for  $k = 0$  and  $k = 1$ .

**Proof:** For  $k \geq 2$ , the formula is obtained by first applying Theorem 5.11 for  $m_1 = 1$ ,  $m_2 = j$ ,  $m_3 = k - j$ , and summing over  $j = 1, \dots, k - 1$ , then using the genus-2  $L_{-1}$  and  $L_0$  constraints:

$$\psi_0 = -A_1(\tau_-^2(\mathcal{S})) \quad \text{and} \quad \psi_1 = A_1(\tau_-^2(\mathcal{L}_0)).$$

For  $k = 1$ , the formula is trivial. For  $k = 0$ , it follows from Theorem 5.11 for  $m_1 = 1$ ,  $m_2 = 0$ ,  $m_3 = 0$ .  $\square$

**Remark:** The right hand side of the formula in Corollary 5.12 only depends on genus-0 and genus-1 data. Therefore once  $\psi_2$  is known, we can compute  $\psi_k$  recursively from this formula for all  $k \geq 3$ . This is the main reason why we should expect the result in Theorem 5.9.

## 5.4 Solving the genus-2 generating function

In this subsection, we will show that if the quantum cohomology is not too degenerate, then the recursive relation in Corollary 5.12 not only determines all functions  $\psi_k$ , but also determines the genus-2 generating function  $F_2$ . In fact, we can give a formula for  $F_2$  in terms of genus-0 and genus-1 data.

Since at each point, the space of primary vectors is finite dimensional, in a neighborhood of a generic point, there exists an integer  $n$  such that  $\{\overline{\mathcal{X}}^k \mid k = 0, \dots, n\}$  are linearly independent and

$$\overline{\mathcal{X}}^{n+1} = \sum_{i=0}^n f_i \overline{\mathcal{X}}^i \quad (52)$$

where  $f_i$  are functions in an open subset of the big phase space. Multiplying both sides by  $\overline{\mathcal{X}}^k$ , we have

$$\overline{\mathcal{X}}^{n+1+k} = \sum_{i=0}^n f_i \overline{\mathcal{X}}^{i+k} \quad (53)$$

for any  $k \geq 0$ .

**Lemma 5.13**

$$T(\mathcal{W})f_i = 0$$

for any vector field  $\mathcal{W}$  and  $i = 0, \dots, n$ .

**Proof:** Taking derivative of equation (52) along the direction  $T(\mathcal{W})$  and using Lemma 3.4, we have

$$-\overline{\mathcal{X}}^{n+1} \bullet \mathcal{W} = \sum_{i=0}^n \{T(\mathcal{W})f_i\} \overline{\mathcal{X}}^i - \sum_{i=0}^n f_i \overline{\mathcal{X}}^i \bullet \mathcal{W}.$$

The lemma then follows from equation (52).  $\square$

Define a sequence of vector fields

$$\mathcal{Y}_k := \sum_{i=0}^{k-1} \overline{\mathcal{X}}^i \bullet ((\mathcal{G} - \frac{1}{2}\mathcal{Z}) * \overline{\mathcal{X}}^{k-1-i}) \quad (54)$$

where  $\mathcal{Z}$  is defined by equation (33). As explained in [L1] [L2], in order for the genus-1 Virasoro conjecture to hold, the sequence of genus-0 functions representing  $\langle\langle \overline{\mathcal{X}}^k \rangle\rangle_1$  as predicted by the genus-1 Virasoro conjecture must be compatible with the linear relation (52) (this condition was called the algebraic compatibility condition in [L2]). Considering this fact and the precise formula for  $\langle\langle \overline{\mathcal{X}}^k \rangle\rangle_1$  as given in [L1], it is reasonable to make the following *assumption*:

$$\mathcal{Y}_{n+1} = \sum_{i=0}^n f_i \mathcal{Y}_i. \quad (55)$$

This assumption is satisfied for all manifolds with semisimple quantum cohomology (cf. [L2]). Moreover this assumption implies the algebraic compatibility condition for the genus-1 Virasoro conjecture. To see this, we define for any vector field  $\mathcal{W}$ ,

$$y_k(\mathcal{W}) = \sum_{i=0}^{k-1} \overline{\mathcal{X}}^i \bullet ((\mathcal{G} - \frac{1}{2}\mathcal{Z}) * (\mathcal{W} \bullet \overline{\mathcal{X}}^{k-1-i})).$$

Then  $y_k(\mathcal{W}) = -\nabla_{T(\mathcal{W})}\mathcal{Y}_k$  by Corollary 1.7, Lemma 3.4 and equation (37). Therefore taking covariant derivative with respect to  $T(\mathcal{W})$  on both sides of equation (55) and using Lemma 5.13, we obtain

$$y_{n+1}(\mathcal{W}) = \sum_{i=0}^n f_i y_i(\mathcal{W}). \quad (56)$$

Moreover, since

$$\mathcal{Y}_{k+1} = y_k(\overline{\mathcal{X}}) + \overline{\mathcal{X}}^k \bullet ((\mathcal{G} - \frac{1}{2}\mathcal{Z}) * \overline{\mathcal{S}}),$$

equation (56) and equation (53) imply that

$$\mathcal{Y}_{n+1+k} = \sum_{i=0}^n f_i \mathcal{Y}_{i+k} \quad (57)$$

for all  $k \geq 0$ . Therefore we also have

$$y_{n+1+k}(\mathcal{W}) = \sum_{i=0}^n f_i y_{i+k}(\mathcal{W}) \quad (58)$$

for all  $k \geq 0$ . When restricted to the small phase space, the formula for  $\langle\langle \overline{\mathcal{X}}^k \rangle\rangle_1$  as given in [L1] is a linear combination of the trace of the map  $\mathcal{W} \mapsto \mathcal{W} \bullet \mathcal{Y}_k$  and the trace of the map  $\mathcal{W} \mapsto y_k((\mathcal{G} - \frac{1}{2}\mathcal{Z}) * \mathcal{W})$ . Therefore the genus-1 algebraic compatibility condition is satisfied. However, we will not use the genus-1 Virasoro constraints in this subsection.

Now we come back to the genus-2 Virasoro conjecture. By Corollary 4.9,

$$\overline{\tau_-(\mathcal{L}_{k-1})} + \frac{3}{2}k \overline{\mathcal{X}}^{k-1} = -\overline{\mathcal{X}}^k \bullet \tau_-(\mathcal{S}) - \mathcal{Y}_k. \quad (59)$$

Define

$$\tilde{\psi}_k := \psi_k - \frac{3k}{2} \langle\langle T(\overline{\mathcal{X}}^{k-1}) \rangle\rangle_2.$$

Then by equation (59),

$$\tilde{\psi}_k = \langle\langle \overline{\mathcal{X}}^k \rangle\rangle_2 + \langle\langle T(\overline{\mathcal{X}}^k \bullet \tau_-(\mathcal{S})) \rangle\rangle_2 + \langle\langle T(\mathcal{Y}_k) \rangle\rangle_2.$$

Therefore equation (57) implies

$$\tilde{\psi}_{n+1+k} = \sum_{i=0}^n f_i \tilde{\psi}_{i+k} \quad (60)$$

for every  $k \geq 0$ . Repeatedly applying this equation, we have,

$$\tilde{\psi}_{n+1+k} = \sum_{i=1}^{n+1} b_{k,i} \tilde{\psi}_i \quad (61)$$

where  $b_{k,i}$  is given by the recursion relation

$$b_{k+1,i} = \begin{cases} \sum_{j=0}^{k-1} f_{j+n-k+1} b_{j+1,i} & \text{for } 1 \leq i \leq k, \\ f_{i-k-1} + \sum_{j=0}^{k-1} f_{j+n-k+1} b_{j+1,i} & \text{for } k+1 \leq i \leq n+1 \end{cases}$$

and

$$b_{1,i} = f_{i-1}$$

for  $1 \leq i \leq n+1$ .

**Lemma 5.14** *For every  $k \geq 0$ ,*

$$\sum_{i=0}^n \left( \frac{n+k}{n+k+2} - \frac{i+k-1}{i+k+1} \right) f_i \tilde{\psi}_{i+k+1} = g_k$$

where

$$g_k := \sum_{j=1}^{n+k} \frac{B(\overline{\mathcal{X}}, \overline{\mathcal{X}}^j, \overline{\mathcal{X}}^{n+1+k-j})}{2(n+k+2)} - \sum_{i=0}^n \sum_{j=1}^{i+k-1} \frac{f_i B(\overline{\mathcal{X}}, \overline{\mathcal{X}}^j, \overline{\mathcal{X}}^{i+k-j})}{2(i+k+1)} + \delta_{k,0} \frac{f_0}{2} B(\overline{\mathcal{X}}, \overline{\mathcal{S}}, \overline{\mathcal{S}}).$$

**Proof:** By equation (7),

$$T(\overline{\mathcal{X}}) \langle\langle T(\overline{\mathcal{X}}^{k-1}) \rangle\rangle_2 = \langle\langle T(\overline{\mathcal{X}}) T(\overline{\mathcal{X}}^{k-1}) \rangle\rangle_2 + \langle\langle \nabla_{T(\overline{\mathcal{X}})} T(\overline{\mathcal{X}}^{k-1}) \rangle\rangle_2.$$

By equation (20), Lemma 1.5 and Lemma 3.4,

$$T(\overline{\mathcal{X}}) \langle\langle T(\overline{\mathcal{X}}^{k-1}) \rangle\rangle_2 = 2 \langle\langle T(\overline{\mathcal{X}}^k) \rangle\rangle_2 + A_2(\overline{\mathcal{X}}, \overline{\mathcal{X}}^{k-1}).$$

Hence by Corollary 5.12,

$$T(\overline{\mathcal{X}}) \tilde{\psi}_k = \frac{2(k-1)}{k+1} \tilde{\psi}_{k+1} - 3 \langle\langle T(\overline{\mathcal{X}}^k) \rangle\rangle_2 - h_k \quad (62)$$

where

$$\begin{aligned} h_k := & A_2 \left( \overline{\mathcal{X}}, \frac{3}{2} k \overline{\mathcal{X}}^{k-1} + \overline{\tau_-(\mathcal{L}_{k-1})} \right) - A_2 \left( \overline{\mathcal{X}}^k, \overline{\tau_-(\mathcal{L}_0)} \right) - T(\overline{\mathcal{X}}^k) A_1(\tau_-^2(\mathcal{L}_0)) \\ & + \frac{1}{k+1} \left\{ -\delta_{k,0} B(\overline{\mathcal{X}}, \overline{\mathcal{S}}, \overline{\mathcal{S}}) + \sum_{j=1}^{k-1} B(\overline{\mathcal{X}}, \overline{\mathcal{X}}^j, \overline{\mathcal{X}}^{k-j}) \right\}. \end{aligned} \quad (63)$$

On the other hand, taking derivative of equation (60) along the direction of  $T(\overline{\mathcal{X}})$  and using Lemma 5.13, we have

$$T(\overline{\mathcal{X}}) \tilde{\psi}_{n+k+1} = \sum_{i=0}^n f_i T(\overline{\mathcal{X}}) \tilde{\psi}_{i+k}.$$

Applying equation (62) to both sides of this equation and using equation (60) and equation (53), we obtain

$$\sum_{i=0}^n \left( \frac{n+k}{n+k+2} - \frac{i+k-1}{i+k+1} \right) f_i \tilde{\psi}_{i+k+1} = \frac{1}{2} \left( h_{n+k+1} - \sum_{i=0}^n f_i h_{i+k} \right).$$

Moreover, equations (59), (53) and (57) imply that

$$\frac{1}{2} \left( h_{n+k+1} - \sum_{i=0}^n f_i h_{i+k} \right) = g_k.$$

The lemma is thus proved.  $\square$

Applying equation (61) to replace every  $\tilde{\psi}_k$  with  $k > n+1$  in Lemma 5.14 by linear combinations of  $\tilde{\psi}_1, \dots, \tilde{\psi}_{n+1}$ , we obtain the following

**Corollary 5.15** *For every  $k \geq 0$ ,*

$$\sum_{i=1}^{n+1} c_{k,i} \tilde{\psi}_i = g_k$$

where  $c_{k,i}$  are given by the recursion relations

$$c_{k,i} = \begin{cases} -\frac{2}{n+k+2} b_{k+1,i} + \sum_{j=0}^{k-1} \frac{2}{j+n+2} f_{j+n-k+1} b_{j+1,i} & \text{for } 1 \leq i \leq k, \\ -\frac{2}{n+k+2} b_{k+1,i} + \frac{2}{i} f_{i-k-1} + \sum_{j=0}^{k-1} \frac{2}{j+n+2} f_{j+n-k+1} b_{j+1,i} & \text{for } k+1 \leq i \leq n+1 \end{cases}$$

and

$$c_{0,i} = \left( \frac{n}{n+2} - \frac{i-2}{i} \right) f_{i-1}$$

for  $1 \leq i \leq n+1$ .

The following lemma will be proved in the appendix.

**Lemma 5.16** *The matrix  $(c_{k,i})$  with  $0 \leq k \leq n$  and  $1 \leq i \leq n+1$  is invertible if and only if the matrix*

$$\begin{pmatrix} \overline{\mathcal{X}} f_0, & \overline{\mathcal{X}}^2 f_0, & \dots, & \overline{\mathcal{X}}^{n+1} f_0 \\ \overline{\mathcal{X}} f_1, & \overline{\mathcal{X}}^2 f_1, & \dots, & \overline{\mathcal{X}}^{n+1} f_1 \\ \dots, & \dots, & \dots, & \dots \\ \overline{\mathcal{X}} f_n, & \overline{\mathcal{X}}^2 f_n, & \dots, & \overline{\mathcal{X}}^{n+1} f_n \end{pmatrix}$$

is invertible.

**Remark:** The columns of the last matrix in this lemma are given by the coefficients of representing the vector fields  $Z_1, \dots, Z_{n+1}$  defined in [L1] and [L2] in terms of  $\overline{\mathcal{X}}^0, \dots, \overline{\mathcal{X}}^n$ .

**Theorem 5.17** *Assume that equation (55) holds. If the polynomial*

$$p(x) = x^{n+1} - \sum_{i=0}^n f_i x^i$$

*does not have repeated roots at generic points, then the generating function for genus-2 Gromov-Witten invariants is given by*

$$F_2 = \frac{1}{2}A_1(\tau_-(\mathcal{S})) + \frac{1}{3}A_1(\tau_-^2(\mathcal{L}_0)) - \frac{1}{3} \sum_{k=0}^n \lambda_{1,k} g_k$$

*where  $(\lambda_{i,k})$  is the inverse of the matrix  $(c_{k,i})$  with  $0 \leq k \leq n$  and  $1 \leq i \leq n+1$  and  $g_k$  is defined in Lemma 5.14. Moreover for  $2 \leq i \leq n+1$ ,*

$$\psi_i = (i-1) \sum_{k=0}^n \lambda_{i,k} g_k - \frac{i}{2} T(\overline{\mathcal{X}}) \sum_{k=0}^n \lambda_{i-1,k} g_k - \frac{i}{2} h_{i-1}$$

*where  $h_i$  is defined by equation (63).*

**Proof:** If the polynomial  $p(x)$  does not have repeated roots, then by Lemma 5.16 and [L2, Lemma 3.4], the matrix  $(c_{k,i})$  with  $0 \leq k \leq n$  and  $1 \leq i \leq n+1$  is invertible. Let  $(\lambda_{i,k})$  be the inverse of this matrix. Then by Corollary 5.15,

$$\tilde{\psi}_i = \sum_{k=0}^n \lambda_{i,k} g_k$$

for  $i = 1, \dots, n+1$ . By definition of  $\tilde{\psi}_1$  and the genus-2  $L_0$ -constraint

$$\langle\langle T(\overline{\mathcal{S}}) \rangle\rangle_2 = \frac{2}{3} (\psi_1 - \tilde{\psi}_1) = \frac{2}{3} \left( A_1(\tau_-^2(\mathcal{L}_0)) - \sum_{k=0}^n \lambda_{1,k} g_k \right).$$

Since

$$\mathcal{D} = T(\mathcal{S}) = T(\overline{\mathcal{S}}) + T^2(\tau_-(\mathcal{S})),$$

by the genus-2 dilaton equation and equation (19)

$$F_2 = \frac{1}{2} \langle\langle \mathcal{D} \rangle\rangle_2 = \frac{1}{2} \left\{ \langle\langle T(\overline{\mathcal{S}}) \rangle\rangle_2 + A_1(\tau_-(\mathcal{S})) \right\}.$$

Therefore we obtain the desired formula for  $F_2$ . Moreover, by equation (62), we have

$$\langle\langle T(\overline{\mathcal{X}}^k) \rangle\rangle_2 = -\frac{1}{3} T(\overline{\mathcal{X}}) \tilde{\psi}_k + \frac{2(k-1)}{3(k+1)} \tilde{\psi}_{k+1} - \frac{1}{3} h_k$$

for  $1 \leq k \leq n$ . The formula for  $\psi_i$  is then obtained by using the definition of  $\tilde{\psi}_i$ . The theorem is proved.  $\square$

**Remark:** (1) If the quantum cohomology of the underlying manifold is semisimple, the conditions in Theorem 5.17 are satisfied (cf. [L1] [L2]). So in this case, we obtained an explicit solution for the generating function of genus-2 Gromov-Witten invariants. (2) The conditions in Theorem 5.17 is a sufficient condition. With careful analysis of the equation in Corollary 5.15, one expects that the condition can be weakened to similar conditions as in [L1] and [L2].



## A Appendix: Proof of Lemma 5.16

Define

$$B_{k+1} = \begin{pmatrix} b_{k+1,1} \\ b_{k+1,2} \\ \vdots \\ b_{k+1,n+1} \end{pmatrix}, \quad C_k = \begin{pmatrix} c_{k,1} \\ c_{k,2} \\ \vdots \\ c_{k,n+1} \end{pmatrix}, \quad D_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_0 \\ \vdots \\ f_{n-k} \end{pmatrix}$$

for  $k = 0, \dots, n$ . Let

$$H = \text{Diag}(1, 2, \dots, n+1)$$

be the  $(n+1) \times (n+1)$  diagonal matrix whose diagonal entries are  $1, 2, \dots, n+1$  and

$$H^{-1} = \text{Diag}(1, \frac{1}{2}, \dots, \frac{1}{n+1}).$$

Then  $B_k$  and  $C_k$  satisfy the recursion relation

$$B_{k+1} = D_k + \sum_{i=0}^{k-1} f_{n-i} B_{k-i}$$

and

$$C_k = 2H^{-1}D_k - \frac{2}{n+k+2}B_{k+1} + \sum_{j=0}^{k-1} \frac{2}{j+n+2}f_{j+n-k+1}B_{j+1}. \quad (64)$$

Let

$$M = \begin{pmatrix} -1 & f_n & f_{n-1} & f_{n-2} & \cdots & f_2 & f_1 \\ 0 & -1 & f_n & f_{n-1} & \cdots & f_3 & f_2 \\ 0 & 0 & -1 & f_n & \cdots & f_4 & f_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & f_n \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Then the inverse of  $M$  has the following form

$$M^{-1} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 0 & a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_0 & a_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_0 \end{pmatrix}$$

where  $a_0 = -1$  and for  $1 \leq k \leq n$ ,

$$a_k = \sum_{i=0}^{k-1} a_i f_{n-k+i+1}.$$

Equation (64) implies that

$$\frac{1}{n+1+k}B_k = \sum_{i=0}^{k-1} a_i \left( \frac{1}{2}C_{k-1-i} - H^{-1}D_{k-1-i} \right). \quad (65)$$

Let

$$\begin{aligned} \tilde{C}_0 &= \frac{1}{2}C_0, \\ \tilde{C}_k &= \frac{1}{2}C_k - \frac{1}{2} \sum_{i=0}^{k-1} \left( \sum_{j=i}^{k-1} f_{j+n-k+1} a_{j-i} \right) C_i \end{aligned}$$

for  $1 \leq k \leq n$  and

$$\begin{aligned} V_0 &= (n+2)H\tilde{C}_0, \\ V_k &= H \left\{ (n+k+2)\tilde{C}_k - \sum_{i=1}^k (n+k-i+2)f_{n-i+1}\tilde{C}_{k-i} \right\} \end{aligned}$$

for  $1 \leq k \leq n$ . Then

$$\begin{aligned} &\det(C_0, C_1, \dots, C_n) \neq 0 \\ \Leftrightarrow &\det(\tilde{C}_0, \tilde{C}_1, \dots, \tilde{C}_n) \neq 0 \\ \Leftrightarrow &\det(V_0, V_1, \dots, V_n) \neq 0. \end{aligned} \quad (66)$$

Moreover, using equation (65), we obtain

$$V_k = (n+k+2)D_k - H \cdot D_k - \sum_{j=0}^{k-1} p_{k-j}D_j \quad (67)$$

where

$$p_k := \sum_{i=1}^k i f_{n-i+1} a_{k-i}.$$

On the other hand, the recursion relation in [L2, Lemma 2.3] can be written as

$$\overline{\mathcal{X}} D_k = (n+k+2)D_k - H D_k, \quad (68)$$

and

$$\overline{\mathcal{X}}^k D_0 = (\overline{\mathcal{X}}^{k-1} f_n) D_0 + \overline{\mathcal{X}}^{k-1} D_1.$$

Since  $D_k$  is obtained from  $D_0$  by a simple shift, recursively applying this formula, we obtain

$$\overline{\mathcal{X}}^k D_0 = \sum_{i=0}^{k-2} (\overline{\mathcal{X}}^{k-1-i} f_n) D_i + \overline{\mathcal{X}} D_{k-1}. \quad (69)$$

**Lemma A.1**

$$\overline{\mathcal{X}}^k f_n = -p_k$$

for  $k \geq 1$ .

**Proof:** The lemma is true for  $k = 1$  since  $\overline{\mathcal{X}} f_n = f_n = -p_1$ . Assume the lemma holds for  $1 \leq k \leq m$ . By equation (69),

$$\overline{\mathcal{X}}^{m+1} f_n = \overline{\mathcal{X}} f_{n-m} + \sum_{i=0}^{m-1} f_{n-i} \overline{\mathcal{X}}^{m-i} f_n.$$

By the induction hypothesis and equation (68),

$$\begin{aligned} \overline{\mathcal{X}}^{m+1} f_n &= (m+1)f_{n-m} - \sum_{i=0}^{m-1} f_{n-i} \sum_{j=1}^{m-i} j f_{n-j+1} a_{m-i-j} \\ &= (m+1)f_{n-m} - \sum_{j=1}^m j f_{n-j+1} \sum_{i=0}^{m-j} f_{n-i} a_{m-i-j}. \end{aligned}$$

By the recursion relation for  $a_k$ ,

$$\overline{\mathcal{X}}^{m+1} f_n = (m+1)f_{n-m} - \sum_{j=1}^m j f_{n-j+1} a_{m-j+1} = -p_{m+1}.$$

So the lemma is proved by induction on  $k$ .  $\square$

Lemma A.1 and equation (67), (68), (69) implies that

$$V_k = \overline{\mathcal{X}}^{k+1} D_0$$

for  $0 \leq k \leq n$ . Therefore Lemma 5.16 follows from equation (66).

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